

Polish spaces of causal curves

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Abstract

We propose and study a new approach to the topologization of spaces of (possibly not all) future-directed causal curves in a stably causal spacetime. It relies on parametrizing the curves “in accordance” with a chosen time function. Thus obtained topological spaces of causal curves are separable and completely metrizable, i.e. Polish. The latter property renders them particularly useful in the optimal transport theory. To illustrate this fact, we explore the notion of a causal time-evolution of measures in globally hyperbolic spacetimes and discuss its physical interpretation.

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1 Introduction

The notion of a causal curve is one of the central concepts in mathematical relativity. Causal curves not only model the worldlines of physical particles, but also determine the causal structure of a given spacetime. It is therefore not surprising that the topological properties of (the particular subsets of) the set of all causal curves can provide an insight into the structure of a given spacetime. Most notably, the historically first definition of global hyperbolicity due to Leray [22] (later adopted by Geroch [18]) invoked the compactness of the set of all causal curves linking two distinct events. Since then, various topologizations of spaces of causal curves were considered, each of them having its advantages and disadvantages. The two most popular approaches lead through the so-called C^0 -topology [18, 19, 27], and through the compact-open topology on the space of causal curves parametrized by their arc-length [11, 23]. See also [29, 30] for more details.

In the present paper, we put forward a new approach to the topologization of spaces of (possibly not all) future-directed causal curves in a given stably causal spacetime \mathcal{M} . In short, it relies on considering the set of causal curves parametrized “in accordance” with a chosen time function $\mathcal{T} : \mathcal{M} \rightarrow \mathbb{R}$, and endowing this set with the compact-open topology. The main advantage of this approach lies in the exceptionally good topological properties of thus obtained space, summarized by the adjective: Polish.

Polish spaces are separable and completely metrizable topological spaces. Their properties render them extremely useful in probability theory [17], optimal transport theory

[2, 3] and mathematical logic (the so-called descriptive set theory is founded on them [20]). Basic examples of Polish spaces include (second countable topological) manifolds, separable Banach spaces, as well as certain spaces of continuous functions. What is more, given a Polish space \mathcal{Y} , the space $\mathcal{P}(\mathcal{Y})$ of all Borel probability measures on \mathcal{Y} (endowed with a suitable topology) is Polish itself.

In the context of mathematical relativity, it was already Geroch who noted that the space of causal curves connecting two distinct events equipped with the C^0 -topology is separable and metrizable [18]. However, the important question of its completeness has not been addressed.

The Polish spaces of causal curves we introduce in this paper turn out to have a natural application in the Lorentzian version of the optimal transport theory. The latter is a fast developing area of research [8, 9, 31], which e.g. opened up a novel approach to the early universe reconstruction problem [10, 15, 16].

In our previous work [13] we extended the standard causal precedence relation \preceq between events of a given spacetime \mathcal{M} onto the space of Borel probability measures on \mathcal{M} and studied its properties. Building upon these results, in the current paper we investigate the notion of the *causal time-evolution of measures* in globally hyperbolic spacetimes. Concretely, suppose one is given a measure-valued map $t \mapsto \mu_t$ with all μ_t 's supported on the level sets of a chosen Cauchy temporal function \mathcal{T} . Call such a measure-valued map *causal* if $\mu_s \preceq \mu_t$ whenever $s \leq t$. We prove that the latter condition is actually equivalent to the existence of a certain (possibly non-unique) measure σ on a suitable Polish space of causal curves. Particular μ_t 's can be retrieved from σ as the pushforward measures $\mu_t = (\text{ev}_t)_\# \sigma$.

The outline of the paper is as follows: In Section 2 we briefly recall some basic definitions and results of causality theory and the theory of Polish spaces needed in the paper. In Section 3 we introduce and study the Polish spaces of causal curves announced above, showing, in particular, the deep relationship between their topology and the C^0 -topology as presented in [27]. In contrast to Section 3, which is embedded within the standard Lorentzian geometry, in the remaining two sections we apply the Polish spaces of causal curves to the Lorentzian optimal transport theory. Section 4 starts with a brief recollection of the causal precedence relation \preceq between Borel probability measures. We then state, in the form of Theorem 5, the equivalence discussed in the previous paragraph. In the course of the proof we develop some additional tools and results on the verge of Lorentzian geometry and optimal transport theory. Finally, Section 5 is devoted to the physical interpretation of Theorem 5 in the context of mathematical relativity. More concretely, we explore the phenomenon of the causal time-evolution of physical quantities distributed in space as seen by different observers.

2 Preliminaries

Throughout the paper we adopt the convention that $\mathbb{N} := \{1, 2, \dots\}$. The set-theoretical inclusion \subseteq is assumed reflexive and its complement $\not\subseteq$ is not to be confused with \subsetneq , which stands for the strict inclusion. The set-theoretical complement of a set \mathcal{X} is denoted by \mathcal{X}^c , whereas its topological closure and interior by $\overline{\mathcal{X}}$ and $\text{int } \mathcal{X}$, respectively. Furthermore, $\mathcal{B}(\mathcal{X})$ denotes the σ -algebra of Borel subsets of the topological space \mathcal{X} .

The space of continuous, continuous and bounded, continuous and compactly supported real-valued functions on a topological space \mathcal{Y} will be respectively denoted by $C(\mathcal{Y})$, $C_b(\mathcal{Y})$, $C_c(\mathcal{Y})$. Analogous spaces of smooth functions on a smooth manifold \mathcal{M} will be respectively denoted by $C^\infty(\mathcal{M})$, $C_b^\infty(\mathcal{M})$, $C_c^\infty(\mathcal{M})$.

Finally, π^i ($i \in \mathbb{N}$) denotes the canonical projection map on the i -th coordinate, the symbol \cong always stands for ‘is homeomorphic to’, whereas $I \subseteq \mathbb{R}$ (sometimes with an additional subscript) will each time denote a nonempty interval.

2.1 Causality theory

In this subsection we recall some definitions and facts from causality theory needed in further investigations. For a detailed exposition of this theory the reader is referred to [4, 24, 26, 27] or to Section 2.3 in [13].

Let \mathcal{M} be a spacetime, i.e. a connected time-oriented Lorentzian manifold with metric denoted by g . For any $p, q \in \mathcal{M}$, we say that p *causally precedes* q , denoted by $p \preceq q$, iff there exists a piecewise smooth future-directed causal curve $\gamma : [0, 1] \rightarrow \mathcal{M}$ from p to q , i.e. $\gamma(0) = p$ and $\gamma(1) = q$.

The relation \preceq is reflexive and transitive. If it is additionally antisymmetric, we call \mathcal{M} a *causal* spacetime. In any case, \prec denote the irreflexive kernel of \preceq .

For any $p \in \mathcal{M}$ one considers the sets $J^+(p), J^-(p) \subseteq \mathcal{M}$, called the *causal future* and *past* of p , respectively. Additionally, for any subset $\mathcal{X} \subseteq \mathcal{M}$ one introduces $J^\pm(\mathcal{X}) := \bigcup_{p \in \mathcal{X}} J^\pm(p)$. Finally, J^+ *tout court* stands for the set of all pairs $(p, q) \in \mathcal{M}^2$ such that $p \preceq q$.

For the definition of a future-directed causal curve to make sense, one needs the curve to be (piecewise) differentiable. Nevertheless, one speaks of causal curves, which are only continuous, in the following sense [23, 24]. A curve $\gamma : I \rightarrow \mathcal{M}$ is *future-directed causal* iff for any convex set $U \subseteq \mathcal{M}$ and any $s, t \in I$, $s < t$ with $\gamma([s, t]) \subseteq U$, it is $\gamma(s) \prec_U \gamma(t)$, where \prec_U denotes the causal precedence relation on U treated as a spacetime on its own right. This definition greatly simplifies under a rather mild assumption on the causal properties of \mathcal{M} . Concretely [24, Prop. 3.19],

Proposition 1. Let \mathcal{M} be a distinguishing spacetime¹. A curve $\gamma : I \rightarrow \mathcal{M}$ is future-directed causal iff $\forall s, t \in I \ s < t \Rightarrow \gamma(s) \prec \gamma(t)$.

For brevity, we shall omit mentioning future-directedness and speak simply about *causal curves*. Following [30], a *causal path* is defined here as an image of a causal curve². In causal spacetimes, a causal path can be equivalently regarded as an equivalence class of causal curves modulo a (continuous and strictly increasing) reparametrization. For this reason, we adopt the notation³ $[\gamma]$ to denote the causal path associated to the causal curve γ . It will always be clear from the context whether one should interpret $[\gamma]$ as a class of curves or as a subset of \mathcal{M} .

For a causal curve $\gamma : I \rightarrow \mathcal{M}$, where $I = [a, b]$ (or $I = [a, b)$, or $I = [a, +\infty)$) we call $\gamma(a)$ the *past endpoint* of γ . One similarly defines the *future endpoint* of a causal curve.

A causal curve $\gamma : (a, b) \rightarrow \mathcal{M}$ ($-\infty \leq a < b \leq +\infty$) is called *past (future) extendible* iff it has a continuous extension onto $[a, b)$ (onto $(a, b]$). In such a case, the point $\lim_{t \searrow a} \gamma(t)$ ($\lim_{t \nearrow b} \gamma(t)$) is also called the past (future) endpoint of γ , analogously as for the curves discussed in the previous paragraph.

¹For the definition of a distinguishing spacetime, see [24, Section 3.2].

²Note that e.g. in [27] the meanings of the terms ‘‘curve’’ and ‘‘path’’ are swapped (which actually is in accordance with the topological terminology).

³Minguzzi and Sánchez in [24] use boldface in this context, but γ will have a different meaning below (see Section 4).

Notice that, in causal spacetimes, if the causal curve γ has a past/future endpoint, then the latter notion is also well defined for the corresponding causal path $[\gamma]$.

A causal curve which is neither past nor future extendible is called *inextendible*. Recall that a *Cauchy hypersurface* is a subset $\mathcal{S} \subseteq \mathcal{M}$ which is met exactly once by every inextendible timelike curve. Any such \mathcal{S} is a closed topological hypersurface, met by every inextendible causal curve [26, Chapter 14, Lemma 29.]. Of course, if \mathcal{S} is additionally spacelike, then the previous sentence can be strengthened by adding “exactly once” at the end.

A function $\mathcal{T} : \mathcal{M} \rightarrow \mathbb{R}$ is called

- a *time function* iff it is continuous and strictly increasing along every future-directed causal curve;
- a *temporal function* iff it is a smooth function with a past-directed timelike gradient.

Every temporal function is a time function, but even a smooth time function need not be temporal.

We now recall three more definitions concerning the causal properties of a spacetime \mathcal{M} , each of them stronger than the preceding one.

\mathcal{M} is called *stably causal* iff it admits a time function⁴. Every stably causal spacetime is distinguishing and causal. Moreover, it admits a temporal function as well.

\mathcal{M} is called *causally simple* iff it is causal and satisfies one of the following equivalent conditions [24, Proposition 3.68]:

- $J^+(p)$ and $J^-(p)$ are closed for every $p \in \mathcal{M}$;
- $J^+(\mathcal{K})$ and $J^-(\mathcal{K})$ are closed for every compact $\mathcal{K} \subset \mathcal{M}$;
- J^+ is a closed subset of \mathcal{M}^2 .

Finally, \mathcal{M} is called *globally hyperbolic* iff it satisfies one of the following equivalent conditions:

- \mathcal{M} is causal and the sets $J^+(p) \cap J^-(q)$ are compact for all $p, q \in \mathcal{M}$;
- \mathcal{M} admits a temporal function \mathcal{T} , the level sets of which are (smooth spacelike) Cauchy hypersurfaces [5].

For short, any time (temporal) function whose level sets are Cauchy hypersurfaces is called a *Cauchy time* (temporal) function.

For the later use, let us present some properties of globally hyperbolic spacetimes.

Proposition 2. Let \mathcal{M} be a globally hyperbolic spacetime. The following subsets of \mathcal{M} are compact:

- i) $J^+(\mathcal{K}_1) \cap J^-(\mathcal{K}_2)$ for any compact $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{M}$,
- ii) $J^\pm(\mathcal{K}) \cap \mathcal{S}$ for any compact $\mathcal{K} \subseteq \mathcal{M}$ and any Cauchy hypersurface $\mathcal{S} \subseteq \mathcal{M}$,
- iii) $J^\pm(\mathcal{K}) \cap J^\mp(\mathcal{S})$ for any compact $\mathcal{K} \subseteq \mathcal{M}$ and any Cauchy hypersurface $\mathcal{S} \subseteq \mathcal{M}$.

⁴This is one of the equivalent definitions of stable causality. For others consult, [24, Section 3.8].

Proof For the proofs of *i*), *ii*) see [28, Lemma 11.5] and [24, Property 4 on p.44], respectively. Here, let us only show the compactness of $J^+(\mathcal{K}) \cap J^-(\mathcal{S})$ (the proof for $J^-(\mathcal{K}) \cap J^+(\mathcal{S})$ is analogous).

Observe that $J^+(\mathcal{K}) \cap J^-(\mathcal{S}) = J^+(\mathcal{K}) \cap J^-(\mathcal{S} \cap J^+(\mathcal{K}))$. Indeed, the inclusion “ \supseteq ” is obvious, whereas in order to prove “ \subseteq ” suppose $r \in J^+(\mathcal{K}) \cap J^-(\mathcal{S})$. It means there exist $p \in \mathcal{K}$ and $q \in \mathcal{S}$ such that $p \preceq r \preceq q$. But then $p \preceq q$ and so q in fact belongs to $\mathcal{S} \cap J^+(\mathcal{K})$.

Notice now that the latter set is compact by *ii*) and hence, on the strength of *i*), the intersection $J^+(\mathcal{K}) \cap J^-(\mathcal{S} \cap J^+(\mathcal{K}))$ is compact as well. \square

Recall that in their seminal 2008 paper [5], Bernal and Sánchez proved the smooth version of Geroch’s splitting theorem. Their method was to construct a *Cauchy temporal function* in any given globally hyperbolic spacetime. Let us stress, however, that given such a function \mathcal{T} , one can always create a smooth splitting “associated” to \mathcal{T} in the sense of the following theorem (see [5] or [6, Remark 3.4]).

Theorem 1. (Geroch, Bernal, Sánchez) *Let (\mathcal{M}, g) be a globally hyperbolic spacetime with metric g and let $\mathcal{T} : \mathcal{M} \rightarrow \mathbb{R}$ be a Cauchy temporal function. Then there exists an isometry $\Phi : \mathcal{M} \rightarrow \mathbb{R} \times \Sigma$, where $\Sigma := \mathcal{T}^{-1}(0)$, $\mathcal{T} = \Phi^*\pi^1$ and*

$$(\Phi^*)^{-1}g = -\beta d\pi^1 \otimes d\pi^1 + \mathcal{G} \quad (1)$$

with $\beta : \mathbb{R} \times \Sigma \rightarrow \mathbb{R}$ a positive smooth function and \mathcal{G} a 2-covariant symmetric tensor field on $\mathbb{R} \times \Sigma$, whose restriction to $\{t\} \times \Sigma$ is a Riemannian metric for every $t \in \mathbb{R}$ and whose radical at each $(t, x) \in \mathbb{R} \times \Sigma$ is spanned by the gradient $(\text{grad } \pi^1)|_{(t,x)}$.

Let us pull formula (1) back on \mathcal{M} . Applying Φ^* to both sides, we can write that

$$g = -\alpha d\mathcal{T} \otimes d\mathcal{T} + \bar{g}, \quad (2)$$

where $\alpha := \Phi^*\beta$ is a positive smooth function on \mathcal{M} and $\bar{g} := \Phi^*\mathcal{G}$ is a 2-covariant symmetric tensor field on \mathcal{M} , whose restriction to $\Phi^{-1}(\{t\} \times \Sigma) = \mathcal{T}^{-1}(t)$ is a Riemannian metric for every $t \in \mathbb{R}$ and whose radical at each $p \in \mathcal{M}$ is spanned by the gradient $(\text{grad } \mathcal{T})|_p$.

Consider the “corresponding” Riemannian metric on \mathcal{M} :

$$w_0 := \alpha d\mathcal{T} \otimes d\mathcal{T} + \bar{g}, \quad (3)$$

where the letter w alludes to the Wick rotation. By the celebrated result of Nomizu and Ozeki [25], there exists a smooth positive function u on \mathcal{M} such that $w := uw_0$ is a *complete* Riemannian metric. For the later use let us observe that

$$w = ug + 2u\alpha d\mathcal{T} \otimes d\mathcal{T}. \quad (4)$$

Needless to say, the distance function associated to w induces the original manifold topology of \mathcal{M} . In the following, this metric will prove extremely useful. In order to distinguish it from the auxiliary metric, we will write $d_w, B_w(p, r), B_w(\mathcal{K}, r)$ for its associated distance function and (generalized) r -balls.

2.2 Polish spaces

A topological space \mathcal{Y} is called *Polish* iff it is separable and completely metrizable. Below we recall these of its basic properties and examples, which we will need throughout the paper. We rely here mostly on [21, Appendix A].

Closed subspaces of Polish spaces are Polish. Moreover, countable products of Polish spaces are Polish as well [21, Proposition A.2.].

Every second-countable locally compact Hausdorff (LCH) space is Polish [21, Example A.9.]. In particular, any smooth manifold (and hence any spacetime) \mathcal{M} is a second-countable LCH space and hence a Polish space. Another important example of a Polish space (which, in general, is not locally compact) is the space of all continuous maps from a second-countable LCH space \mathcal{X} to a Polish space \mathcal{Y} , denoted by $C(\mathcal{X}, \mathcal{Y})$, endowed with the compact-open topology [21, Example A.10.]. In particular, taking $\mathcal{X} := I \subseteq \mathbb{R}$ to be a (non-empty) interval and $\mathcal{Y} := \mathcal{M}$ to be a smooth manifold, we see that the space of all *curves* in \mathcal{M} with domain I is Polish. Since \mathcal{M} is metrizable, the compact-open topology on $C(I, \mathcal{M})$ is nothing but the topology of uniform convergence on compact sets. More precisely, the sequence $(\gamma_n) \subseteq C(I, \mathcal{M})$ converges to $\gamma \in C(I, \mathcal{M})$ iff for every compact $K \subseteq I$ $\sup_{t \in K} d(\gamma_n(t), \gamma(t)) \rightarrow 0$, where d denotes the distance function associated to any auxiliary complete Riemannian metric h on \mathcal{M} . Without loss of generality, one might consider only the compact intervals $K = [a, b]$ for $a, b \in I$ when checking convergence.

With the metric h fixed, $B(p, r)$ will denote the r -ball ($r > 0$) centered at $p \in \mathcal{M}$. Additionally, for any compact $\mathcal{K} \subseteq \mathcal{M}$ and any $r > 0$ we define the *generalized r -ball centered at \mathcal{K}* via $B(\mathcal{K}, r) := \bigcup_{p \in \mathcal{K}} B(p, r)$. Observe that, because \mathcal{K} is compact and hence bounded, therefore $B(\mathcal{K}, r)$ is bounded as well⁵ and hence it is precompact⁶.

Polish spaces have “nice” measure-theoretic properties, which make them particularly useful in probability theory and in optimal transport theory. To see this, we first recall the natural topology on the space $\mathcal{P}(\mathcal{Y})$ of all Borel probability measures on a Polish space \mathcal{Y} , called the *narrow topology*. It is defined as the coarsest topology such that for each $f \in C_b(\mathcal{Y})$ the map $\mathcal{P}(\mathcal{Y}) \ni \mu \mapsto \int_{\mathcal{Y}} f d\mu \in \mathbb{R}$ is continuous. Thus, a sequence $(\mu_n) \subseteq \mathcal{P}(\mathcal{Y})$ is *narrowly* convergent to $\mu \in \mathcal{P}(\mathcal{Y})$ iff $\int_{\mathcal{Y}} f d\mu_n \rightarrow \int_{\mathcal{Y}} f d\mu$ for every $f \in C_b(\mathcal{Y})$.

It can be shown that $\mathcal{P}(\mathcal{Y})$ is a Polish space as well [3, Remark 7.1.7], and hence narrowly convergent sequences fully determine the narrow topology. Moreover, the notions of sequential compactness and compactness are equivalent in this topology.

One of the seminal results of the probability theory on Polish spaces is the Prokhorov theorem [3, Theorem 5.1.3], which states that a subset $\mathcal{A} \subseteq \mathcal{P}(\mathcal{Y})$ is relatively compact (in the narrow topology) iff it is *tight*, i.e. iff

$$\forall \varepsilon > 0 \quad \exists \text{ compact } K_\varepsilon \subseteq \mathcal{Y} \quad \forall \mu \in \mathcal{A} \quad \mu(K_\varepsilon) \geq 1 - \varepsilon.$$

This theorem in particular implies that any Borel probability measure on a Polish space is inner regular, i.e. for any $\mu \in \mathcal{P}(\mathcal{Y})$ and any $E \in \mathcal{B}(\mathcal{Y})$

$$\mu(E) = \sup \{ \mu(\mathcal{K}) \mid \mathcal{K} \subseteq E, \mathcal{K} \text{ compact} \}.$$

For brevity, the term “measure” will from now on always stand for “Borel probability measure”.

⁵The proof is a simple application of the triangle inequality.

⁶Recall that, by the Hopf–Rinow theorem, a Riemannian manifold enjoys the Heine–Borel property iff it is a complete metric space [26].

3 Polish spaces of causal curves

The sets of (possibly not all) continuous causal curves can be topologized in various ways. A seemingly natural way is to endow the set of all causal curves with a fixed domain $I \subseteq \mathbb{R}$ with the compact-open topology induced from $C(I, \mathcal{M})$. However, thus obtained space is “too big”, because various parametrizations of the same causal path are regarded as distinct elements. This problem can be solved by suitably choosing a unique parametrization of each causal path. The standard choice is the arc-length parametrization with respect to an auxiliary Riemannian metric [11, 30] and it can even encompass curves with *different* domains [23]. Unfortunately, this particular choice of parametrization has a serious drawback. Namely, the limit of a sequence of arc-length-parametrized curves is usually not parametrized by its arc-length. Therefore, this space is not closed in $C(I, \mathcal{M})$ and hence it is not Polish.

Another standard approach relies on using the so-called C^0 -topology. In the context of mathematical relativity, it is usually introduced on the space $C(p, q)$ of causal paths with fixed endpoints $p, q \in \mathcal{M}$, where \mathcal{M} is assumed causal or strongly causal [4, 18, 19]. Here, however, we follow the exposition from [27], where the endpoints are not fixed.

Definition 1. Let \mathcal{M} be a strongly causal spacetime and let \mathcal{C} denote the set of all compact causal paths. The C^0 -topology on \mathcal{C} is defined via its base, which consists of the sets $\mathcal{C}_U(P, Q)$ of all compact causal paths contained in U , with past endpoint in P and future endpoint in Q , where P, Q, U are open subsets of \mathcal{M} .

Remark 1. The space $C(p, q)$ mentioned above is nothing but $\mathcal{C}_{\mathcal{M}}(\{p\}, \{q\})$ with the C^0 -topology induced from \mathcal{C} . More generally, one can endow spaces $\mathcal{C}_U(P, Q)$ with P, Q, U any subsets of \mathcal{M} with the C^0 -topology induced from \mathcal{C} .

Remark 2. Observe that the sequence $([\gamma_n]) \subseteq \mathcal{C}$ converges to $[\gamma] \in \mathcal{C}$ iff simultaneously:

- The sequence of past endpoints of $[\gamma_n]$ ’s converges in \mathcal{M} to the past endpoint of $[\gamma]$.
- The sequence of future endpoints of $[\gamma_n]$ ’s converges in \mathcal{M} to the future endpoint of $[\gamma]$.
- For any open $U \supseteq [\gamma]$ it is true that $[\gamma_n] \subseteq U$ for sufficiently large n .

It is common to define the C^0 -convergence of *curves* via the C^0 -convergence of their images. Drawing from the above observation, we say⁷ that the sequence $(\gamma_n) \subseteq C([a, b], \mathcal{M})$ converges to $\gamma \in C([a, b], \mathcal{M})$ in the C^0 -topology iff

- i) $\gamma_n(a) \rightarrow \gamma(a)$ and $\gamma_n(b) \rightarrow \gamma(b)$,
- ii) for any open $U \supseteq \gamma([a, b])$ it is true that $\gamma_n([a, b]) \subseteq U$ for sufficiently large n .

Proposition 3. \mathcal{C} is first countable.

Proof Fix any $[\gamma] \in \mathcal{C}$ with endpoints p, q . We claim that the sets of the form $\mathcal{C}_{B([\gamma], 1/n)}(B(p, 1/n), B(q, 1/n))$, $n \in \mathbb{N}$, constitute the (countable) neighborhood basis of $[\gamma]$.

Firstly, let $P, Q \subseteq \mathcal{M}$ be any open neighborhoods of p, q , respectively. Clearly, $B(p, 1/n) \subseteq P$ and $B(q, 1/n) \subseteq Q$ for sufficiently large n .

⁷[4, Definition 3.33]

Secondly, let $U \subseteq \mathcal{M}$ be any open set containing $[\gamma]$. We show that $B([\gamma], 1/n) \subseteq U$ for n large enough.

By contradiction, assume that there exists an increasing, infinite sequence $(n_k) \subseteq \mathbb{N}$ such that $B([\gamma], 1/n_k) \not\subseteq U$. One can thus construct a sequence $(x_k) \subseteq \mathcal{M} \setminus U$ such that $x_k \in B([\gamma], 1/n_k)$ for all k . Notice that (x_k) is contained in a precompact set $B([\gamma], 1)$ and thus has a subsequence convergent to some $x_\infty \in \mathcal{M} \setminus U$, because the latter set is closed. At the same time, since $x_k \in B([\gamma], 1/n_k)$ for all k , we obtain $x_\infty \in [\gamma] \subseteq U$, which is absurd.

Altogether, we have that for any open $P, Q, U \subseteq \mathcal{M}$ such that $\mathcal{C}_U(P, Q) \ni [\gamma]$ it is true that $\mathcal{C}_{B([\gamma], 1/n)}(B(p, 1/n), B(q, 1/n)) \subseteq \mathcal{C}_U(P, Q)$ for n sufficiently large. \square

It was already Geroch who observed that $C(p, q)$ is separable and metrizable [18] (see also [29]). However, to the author's best knowledge, the question whether \mathcal{C} is Polish has not been addressed. In the following, we show that this is indeed the case (at least in stably causal spacetimes). To this end, we introduce the following new approach to the topologization of the space of causal curves, which employs a *time function* to “canonically” parametrize causal paths.

Definition 2. Let \mathcal{M} be a stably causal spacetime and let $\mathcal{T} : \mathcal{M} \rightarrow \mathbb{R}$ be a time function. Fix an interval $I \subseteq \mathbb{R}$ and define $C_{\mathcal{T}}^I$ to be the set of all causal curves $\gamma : I \rightarrow \mathcal{M}$, along which \mathcal{T} increases at a constant pace, i.e.

$$\gamma \in C_{\mathcal{T}}^I \iff \exists c_\gamma > 0 \quad \forall s, t \in I \quad \mathcal{T}(\gamma(t)) - \mathcal{T}(\gamma(s)) = c_\gamma(t - s) \quad (5)$$

We endow $C_{\mathcal{T}}^I$ with the compact-open topology induced from $C(I, \mathcal{M})$.

In the case when $I = [a, b]$, we additionally introduce for any $P, Q \subseteq \mathcal{M}$ the subspace $C_{\mathcal{T}}^{[a, b]}(P, Q) := C_{\mathcal{T}}^{[a, b]} \cap \text{ev}_a^{-1}(P) \cap \text{ev}_b^{-1}(Q)$.

Remark 3. Observe that the constant c_γ in (5) can be expressed as

$$c_\gamma = \frac{\mathcal{T}(\gamma(b)) - \mathcal{T}(\gamma(a))}{b - a}, \quad (6)$$

where $a, b \in I$, $a \neq b$.

This observation leads to an alternative, equivalent formulation of condition (5), which is sometimes more convenient. Namely,

$$\forall t, a, b \in I, a \neq b \quad \mathcal{T}(\gamma(t)) = \frac{b - t}{b - a} \mathcal{T}(\gamma(a)) + \frac{t - a}{b - a} \mathcal{T}(\gamma(b)). \quad (7)$$

It is not difficult to notice that \mathcal{C} is in bijection with $C_{\mathcal{T}}^{[a, b]}$ for any $a, b \in \mathbb{R}$ and any time function \mathcal{T} . Indeed, the proof amounts to showing that the map $[\cdot] : C_{\mathcal{T}}^{[a, b]} \rightarrow \mathcal{C}$, $\gamma \mapsto [\gamma]$ is injective. This is a direct consequence of the following proposition.

Proposition 4. Let \mathcal{M} be a stably causal spacetime, let $\mathcal{T} : \mathcal{M} \rightarrow \mathbb{R}$ be a time function and fix $a, b \in \mathbb{R}$. Then for any causal curve $\gamma : [a', b'] \rightarrow \mathcal{M}$ there exists a unique map $\lambda : [a, b] \rightarrow [a', b']$ continuous, strictly increasing and such that $\gamma \circ \lambda \in C_{\mathcal{T}}^{[a, b]}$.

Proof Because \mathcal{T} is a time function, the map $\mathcal{T} \circ \gamma : [a', b'] \rightarrow [\mathcal{T}(\gamma(a')), \mathcal{T}(\gamma(b'))]$ is continuous, onto and strictly increasing. Hence $(\mathcal{T} \circ \gamma)^{-1}$ exists and is continuous and

strictly increasing as well. One can easily convince oneself that in order for property (7) to be satisfied for a curve $\gamma \circ \lambda : [a, b] \rightarrow \mathcal{M}$, it is necessary and sufficient to define λ via

$$\forall t \in [a, b] \quad \lambda(t) := (\mathcal{T} \circ \gamma)^{-1} \left(\mathcal{T}(\gamma(a')) + \frac{t-a}{b-a} (\mathcal{T}(\gamma(b')) - \mathcal{T}(\gamma(a'))) \right),$$

which clearly is also continuous and strictly increasing and hence constitutes a well-defined reparametrization map. \square

Corollary 1. For any $\gamma_1, \gamma_2 \in C_{\mathcal{T}}^{[a,b]}$ if $[\gamma_1] = [\gamma_2]$ then $\gamma_1 = \gamma_2$.

Proof The equality $[\gamma_1] = [\gamma_2]$ means that there exists a (continuous and strictly increasing) reparametrization map $\lambda : [a, b] \rightarrow [a, b]$ such that $\gamma_1 = \gamma_2 \circ \lambda$. Observe then that $\gamma_2 \circ \lambda \in C_{\mathcal{T}}^{[a,b]}$, but also, trivially, $\gamma_2 \circ \text{id}_{[a,b]} \in C_{\mathcal{T}}^{[a,b]}$. By Proposition 4 we obtain $\lambda = \text{id}_{[a,b]}$ and so $\gamma_1 = \gamma_2$. \square

What is less straightforward is that \mathcal{C} is actually *homeomorphic* to $C_{\mathcal{T}}^{[a,b]}$. Since the former space is first-countable (Proposition 3) and the latter space is metrizable, therefore their topologies are fully determined by the convergent sequences. Their homeomorphicity thus results from the following theorem.

Theorem 2. Let \mathcal{M} be a stably causal spacetime, let $\mathcal{T} : \mathcal{M} \rightarrow \mathbb{R}$ be a time function and fix $a, b \in \mathbb{R}$. Finally, let $(\gamma_n) \subseteq C_{\mathcal{T}}^{[a,b]}$ and $\gamma \in C_{\mathcal{T}}^{[a,b]}$. Then $\gamma_n \rightarrow \gamma$ in $C_{\mathcal{T}}^{[a,b]}$ (e.g. uniformly) iff $[\gamma_n] \rightarrow [\gamma]$ in the C^0 -topology.

Proof (\Rightarrow) It is enough to show that conditions *i*), *ii*) from Remark 2 are satisfied. Because uniform convergence implies pointwise convergence, condition *i*) follows trivially. In order to show *ii*), take any open $U \supseteq [\gamma]$. From the proof of Proposition 3, we already know that $B([\gamma], 1/k) \subseteq U$ for k sufficiently large.

Fix such k and notice now that, because $\gamma_n \rightarrow \gamma$ uniformly, then $[\gamma_n] \subseteq B([\gamma], 1/k)$ for sufficiently large n . Indeed, the former condition means that $\forall t \in [a, b] \quad d(\gamma_n(t), \gamma(t)) < 1/k$ for sufficiently large n , whereas the latter condition is weaker, saying that $\forall t \in [a, b] \quad \exists s \in [a, b] \quad d(\gamma_n(t), \gamma(s)) < 1/k$ for sufficiently large n .

Altogether, we have thus obtained that $[\gamma_n] \subseteq U$ for n sufficiently large.

(\Leftarrow) As the first step, we need the following technical result:

$$\begin{aligned} \forall \varepsilon > 0 \quad \exists \varepsilon', \varepsilon'' > 0 \quad \forall t \in [a, b] \\ \mathcal{T}^{-1}((\mathcal{T}(\gamma(t)) - \varepsilon', \mathcal{T}(\gamma(t)) + \varepsilon')) \cap B([\gamma], \varepsilon'') \subseteq B(\gamma(t), \varepsilon) \end{aligned} \quad (8)$$

Suppose (8) is not true, e.g. one can fix $\varepsilon > 0$ such that for any $\varepsilon', \varepsilon'' > 0$ there exists $t \in [a, b]$ for which the above inclusion does not hold. Let us fix $\varepsilon' := c_{\gamma}\delta$, where c_{γ} is given by (6) and $\delta > 0$ is such that

$$\forall \tau_1, \tau_2 \in [a, b] \quad |\tau_1 - \tau_2| < \delta \Rightarrow d(\gamma(\tau_1), \gamma(\tau_2)) < \varepsilon. \quad (9)$$

Notice that the existence of such δ results directly from the uniform continuity of γ (which property is in turn guaranteed by the Heine–Borel theorem). For any $k \in \mathbb{N}$ we also take $\varepsilon'' := 1/k$ and thus create a sequence $(t_k) \subseteq [a, b]$ such that

$$\mathcal{T}^{-1}((\mathcal{T}(\gamma(t_k)) - c_{\gamma}\delta, \mathcal{T}(\gamma(t_k)) + c_{\gamma}\delta)) \cap B([\gamma], 1/k) \not\subseteq B(\gamma(t_k), \varepsilon)$$

for every k . This, in turn, means that one can construct a sequence $(p_k) \subseteq B([\gamma], 1)$ such that

$$|\mathcal{T}(p_k) - \mathcal{T}(\gamma(t_k))| < c_\gamma \delta \quad \wedge \quad p_k \in B([\gamma], 1/k) \quad \wedge \quad d(p_k, \gamma(t_k)) \geq \varepsilon \quad (10)$$

for every k . Because $[a, b]$ is compact and $B([\gamma], 1)$ is precompact, we can pass to subsequences of (t_k) and (p_k) that (simultaneously) converge to some $t_\infty \in [a, b]$ and $p_\infty \in \overline{B([\gamma], 1)}$, respectively. From (10) we obtain that in fact $p_\infty \in [\gamma]$ and so $p_\infty = \gamma(s)$ for some $s \in [a, b]$. Furthermore, (10) yields also that

$$|\mathcal{T}(\gamma(s)) - \mathcal{T}(\gamma(t_\infty))| < c_\gamma \delta \quad \wedge \quad d(\gamma(s), \gamma(t_\infty)) \geq \varepsilon.$$

However, using (5) we can rewrite the left-hand side of the first condition as $c_\gamma |s - t_\infty|$ and obtain that

$$|s - t_\infty| < \delta \quad \wedge \quad d(\gamma(s), \gamma(t_\infty)) \geq \varepsilon,$$

which contradicts (9) and thus completes the proof of (8).

We are now ready to prove that $\gamma_n \rightarrow \gamma$ uniformly, i.e. that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N \forall t \in [a, b] \quad d(\gamma(t), \gamma_n(t)) < \varepsilon \quad (11)$$

To this end, firstly observe that condition *i)* from Remark 2 implies that $\mathcal{T} \circ \gamma_n \rightarrow \mathcal{T} \circ \gamma$ uniformly, i.e.

$$\forall \varepsilon' > 0 \exists N' \in \mathbb{N} \forall n \geq N' \forall t \in [a, b] \quad |\mathcal{T}(\gamma(t)) - \mathcal{T}(\gamma_n(t))| < \varepsilon'. \quad (12)$$

Indeed, employing (7), we easily get that for any $t \in [a, b]$

$$|\mathcal{T}(\gamma(t)) - \mathcal{T}(\gamma_n(t))| \leq |\mathcal{T}(\gamma(a)) - \mathcal{T}(\gamma_n(a))| + |\mathcal{T}(\gamma(b)) - \mathcal{T}(\gamma_n(b))|,$$

what on the strength of *i)* already yields the uniform convergence.

Additionally, observe that condition *ii)* from Remark 2 implies that

$$\forall \varepsilon'' > 0 \exists N'' \in \mathbb{N} \forall n \geq N'' \quad [\gamma_n] \subseteq B([\gamma], \varepsilon''). \quad (13)$$

Indeed, simply take $U := B([\gamma], \varepsilon'')$.

To prove (11), fix any $\varepsilon > 0$, take $\varepsilon', \varepsilon'' > 0$ as in (8), then take $N', N'' \in \mathbb{N}$ as in (12,13) and define $N := \max\{N', N''\}$. For any $n \geq N$, (12,13) imply that for all $t \in [a, b]$

$$\gamma_n(t) \in \mathcal{T}^{-1}((\mathcal{T}(\gamma(t)) - \varepsilon', \mathcal{T}(\gamma(t)) + \varepsilon')) \cap B([\gamma], \varepsilon''),$$

which, on the strength of (8), gives that $\gamma_n(t) \in B(\gamma(t), \varepsilon)$ for all $t \in [a, b]$, completing the proof of (11). \square

Two straightforward corollaries follow.

Corollary 2. Let $\mathcal{M}, \mathcal{T}, a, b$ be as above and let P, Q be *any* subsets of \mathcal{M} . Then $\mathcal{C}_{\mathcal{M}}(P, Q) \cong C_{\mathcal{T}}^{[a, b]}(P, Q)$.

Proof Because we are only putting constraints on the location of endpoints, the restricted map $[\cdot] : C_{\mathcal{T}}^{[a, b]}(P, Q) \rightarrow \mathcal{C}_{\mathcal{M}}(P, Q)$ is still a well-defined bijection. Moreover, elementary properties of the subspace topology guarantee that it is still a homeomorphism. \square

Corollary 3. Let P, Q be subsets of a stably causal spacetime \mathcal{M} , let $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{M} \rightarrow \mathbb{R}$ be time functions and fix $a_i, b_i \in \mathbb{R}$, $a_i < b_i$ for $i = 1, 2$. Then the spaces $C_{\mathcal{T}_1}^{[a_1, b_1]}(P, Q)$ and $C_{\mathcal{T}_2}^{[a_2, b_2]}(P, Q)$ are homeomorphic.

Proof By the previous corollary, $C_{\mathcal{T}_1}^{[a_1, b_1]}(P, Q) \cong \mathcal{C}_{\mathcal{M}}(P, Q) \cong C_{\mathcal{T}_2}^{[a_2, b_2]}(P, Q)$. \square

Furthermore, Corollary 2 allows to restate Leray's characterization⁸ of global hyperbolicity in terms of the spaces $C_{\mathcal{T}}^{[a, b]}$.

Proposition 5. Let \mathcal{M} be a stably causal spacetime and let $\mathcal{T} : \mathcal{M} \rightarrow \mathbb{R}$ be a time function. Then the following conditions are equivalent.

- i) \mathcal{M} is globally hyperbolic.
- ii) $C_{\mathcal{T}}^{[a, b]}(K_1, K_2)$ is compact for all $a, b \in \mathbb{R}$ and all compact $K_1, K_2 \subseteq \mathcal{M}$.
- iii) $C_{\mathcal{T}}^{[0, 1]}(p, q)$ is compact for all $p, q \in \mathcal{M}$.

Proof With Corollary 2 guaranteeing that $C_{\mathcal{T}}^{[0, 1]}(p, q) \cong C(p, q)$, the equivalence $i) \Leftrightarrow iii)$ follows from [18] (compare also [19, Proposition 6.6.2] and [24, Theorem 3.79]). The implication $i) \Leftarrow ii)$ can be proven by first invoking Corollary 2 to get $C_{\mathcal{T}}^{[a, b]}(K_1, K_2) \cong \mathcal{C}_{\mathcal{M}}(K_1, K_2) = \mathcal{C}_{J^+(K_1) \cap J^-(K_2)}(K_1, K_2)$ and then noticing that the latter space is compact on the strength of [27, Theorem 6.5] and Proposition 2. Finally, the implication $ii) \Leftarrow iii)$ is trivial. \square

We now address the question of Polishness of $C_{\mathcal{T}}^I$.

Proposition 6. Let \mathcal{M} be a stably causal spacetime, let $\mathcal{T} : \mathcal{M} \rightarrow \mathbb{R}$ be a time function and let $I \subseteq \mathbb{R}$ be an interval. Then $C_{\mathcal{T}}^I$ is a Polish space. Moreover, if $I = [a, b]$ then the space $C_{\mathcal{T}}^{[a, b]}(P, Q)$ is Polish for any *closed* $P, Q \subseteq \mathcal{M}$.

Proof We show below that $C_{\mathcal{T}}^I$ is a closed subspace of the Polish space $C(I, \mathcal{M})$. The Polishness of $C_{\mathcal{T}}^{[a, b]}(P, Q)$ for any closed $P, Q \subseteq \mathcal{M}$ will then follow from the continuity of the evaluation maps ev_a, ev_b .

To this end, suppose that $(\gamma_n) \subseteq C_{\mathcal{T}}^I$ converges to $\gamma \in C(I, \mathcal{M})$ in the compact-open topology. In order to show that γ satisfies (7), fix any $t, a, b \in I$, $a \neq b$ and use the fact that the uniform convergence on compact sets implies the pointwise convergence to obtain

$$\begin{aligned} \mathcal{T}(\gamma(t)) &= \lim_{n \rightarrow +\infty} \mathcal{T}(\gamma_n(t)) = \lim_{n \rightarrow +\infty} \frac{b-t}{b-a} \mathcal{T}(\gamma_n(a)) + \frac{t-a}{b-a} \mathcal{T}(\gamma_n(b)) \\ &= \frac{b-t}{b-a} \mathcal{T}(\gamma(a)) + \frac{t-a}{b-a} \mathcal{T}(\gamma(b)). \end{aligned}$$

We still have to prove that γ is future-directed and causal. To achieve this, we will use Proposition 1.

Let us define two sequences $(a_m), (b_m) \subseteq I$, $a_m < b_m$ for all $m \in \mathbb{N}$ as follows. If $\inf I \in I$, define $a_m \equiv \inf I$, otherwise let (a_m) be a decreasing sequence tending to $\inf I$. Similarly, if $\sup I \in I$, define $b_m \equiv \sup I$, otherwise let (b_m) be an increasing sequence tending to $\sup I$.

⁸Historically, this actually was the very definition of global hyperbolicity.

By assumption, for every $m \in \mathbb{N}$ the sequence $(\gamma_n|_{[a_m, b_m]})_{n \in \mathbb{N}}$ converges to $\gamma|_{[a_m, b_m]}$ uniformly, and hence also in the C^0 -topology by Theorem 2. On the strength of [4, Proposition 3.34 & Lemma 3.29], the latter convergence implies that $\gamma|_{[a_m, b_m]}$ is future-directed causal for every $m \in \mathbb{N}$. This, in turn, clearly shows that γ satisfies the characterization given in Proposition 1, because any $s, t \in I$ lie in $[a_m, b_m]$ for m large enough. \square

As an immediate corollary of Theorem 2 (with Corollary 3) and Proposition 6 we obtain:

Corollary 4. Let \mathcal{M} be a stably causal spacetime. Then for any closed $P, Q \subseteq \mathcal{M}$ the space $\mathcal{C}_{\mathcal{M}}(P, Q)$ is Polish. In particular, \mathcal{C} is a Polish space.

We see that the spaces $C_{\mathcal{T}}^I$ with $I = [a, b]$ cast some new light on \mathcal{C} , i.e. the space of all *compact* causal paths. But what about *noncompact* causal paths?

Below we provide a “noncompact” analogue (or rather a complement) of Proposition 4, which answers the question whether a *noncompact* causal path can be uniquely parametrized so as to become an element of $C_{\mathcal{T}}^I$ for a suitable interval I . The answer turns out to be rather subtle, heavily depending on the time function \mathcal{T} employed. Namely, if \mathcal{T} is *bounded* on a given causal path, such a unique parametrization exists. If, on the other hand, \mathcal{T} is unbounded on a causal path, the parametrization of the latter is unique only up to a certain affine transformation.

Proposition 7. Let \mathcal{M} be a stably causal spacetime and let $\mathcal{T} : \mathcal{M} \rightarrow \mathbb{R}$ be a time function. Let $I' := (a', b')$, $-\infty \leq a' < b' \leq +\infty$ and let $\gamma : I' \rightarrow \mathcal{M}$ be a causal curve. The map $\mathcal{T} \circ \gamma$ is continuous, strictly increasing and hence onto (T_{γ}, T^{γ}) , where $T_{\gamma} := \lim_{\tau \searrow a'} \mathcal{T}(\gamma(\tau))$ and $T^{\gamma} := \lim_{\tau \nearrow b'} \mathcal{T}(\gamma(\tau))$ might be equal to $-\infty$ and $+\infty$, respectively. One has that:

- If both T_{γ} and T^{γ} are finite, then $\gamma \circ \lambda \in C_{\mathcal{T}}^I$ if and only if $I = (a, b)$, $a, b \in \mathbb{R}$ and $\lambda : I \rightarrow I'$ is of the form

$$\forall t \in I \quad \lambda(t) := (\mathcal{T} \circ \gamma)^{-1} \left(T_{\gamma} + \frac{t - a}{b - a} (T^{\gamma} - T_{\gamma}) \right).$$

The same is true if $I' = [a', b']$ ($I' = (a', b']$), only now $I = [a, b]$ ($I = (a, b]$).

- If T_{γ} is finite and $T^{\gamma} = +\infty$, then $\gamma \circ \lambda \in C_{\mathcal{T}}^I$ if and only if $I = (a, +\infty)$, $a \in \mathbb{R}$ and $\lambda : I \rightarrow I'$ is of the form

$$\forall t \in I \quad \lambda(t) := (\mathcal{T} \circ \gamma)^{-1} (T_{\gamma} + A(t - a)),$$

where A is an arbitrary positive constant. The same is true if $I' = [a', b')$, only now $I = [a, +\infty)$.

- If $T_{\gamma} = -\infty$ and T^{γ} is finite, then $\gamma \circ \lambda \in C_{\mathcal{T}}^I$ if and only if $I = (-\infty, b)$, $b \in \mathbb{R}$ and $\lambda : I \rightarrow I'$ is of the form

$$\forall t \in I \quad \lambda(t) := (\mathcal{T} \circ \gamma)^{-1} (T^{\gamma} + A(t - b)),$$

where A is an arbitrary positive constant. The same is true if $I' = (a', b]$, only now $I = (-\infty, b]$.

- If $T_\gamma = -\infty$ and $T^\gamma = +\infty$, then $\gamma \circ \lambda \in C_{\mathcal{T}}^I$ if and only if $I = \mathbb{R}$ and $\lambda : \mathbb{R} \rightarrow I'$ is of the form

$$\forall t \in \mathbb{R} \quad \lambda(t) := (\mathcal{T} \circ \gamma)^{-1}(At + B),$$

where A, B are arbitrary real constants with $A > 0$.

Proof That $\mathcal{T} \circ \gamma$ is continuous and strictly increasing results from the very definition of a time function. Hence, the inverse map $(\mathcal{T} \circ \gamma)^{-1} : (T_\gamma, T^\gamma) \rightarrow I'$ exists, it is continuous and strictly increasing.

Similarly as in the proof of Proposition 4, one can easily convince himself that in each particular case, the given formula for λ is necessary and sufficient for the curve $\gamma \circ \lambda$ to satisfy (5) under the assumptions pertaining to that case.

Finally, notice that if $I' = [a', b')$, then one has to replace (T_γ, T^γ) with $[T_\gamma, T^\gamma)$, and also define I as containing its left endpoint. Similar simple modifications are needed in the $I' = (a', b']$ case. \square

As we can see, if the interval I is noncompact, then the question which causal paths are “included” in $C_{\mathcal{T}}^I$ and which are not is rather complicated, depending on the details of the chosen time function \mathcal{T} and the interval I . In particular, there is no simple analogue of Corollary 3. Although addressing these questions in full generality lies beyond the scope of this paper, below we answer them assuming some additional properties of \mathcal{T} . We shall need these results for the application discussed in the paper’s further sections.

Proposition 8. Let \mathcal{M} be a globally hyperbolic spacetime and let $\mathcal{T} : \mathcal{M} \rightarrow \mathbb{R}$ be a *Cauchy* time function. Then the map $C_{\mathcal{T}}^{\mathbb{R}} \ni \gamma \mapsto [\gamma]$ is a surjection onto the set of all inextendible causal paths on \mathcal{M} .

Proof We need to show two things: that every $\gamma \in C_{\mathcal{T}}^{\mathbb{R}}$ is an inextendible causal curve (and hence $[\gamma]$ is an inextendible causal path) and that every inextendible causal path can be parametrized so as to become an element of $C_{\mathcal{T}}^{\mathbb{R}}$. In fact, the latter claim is a direct consequence of Proposition 7 (the last bullet), because \mathcal{T} , being a Cauchy time function, assumes all real values on every inextendible causal path⁹.

As for the first claim, suppose that $\gamma \in C_{\mathcal{T}}^{\mathbb{R}}$ has a future endpoint $q := \lim_{t \rightarrow +\infty} \gamma(t)$. By (5), we can write that

$$\forall t \in \mathbb{R} \quad \mathcal{T}(\gamma(t)) = \mathcal{T}(\gamma(0)) + c_\gamma t.$$

But passing now with t to $+\infty$ and using the continuity of \mathcal{T} , we obtain

$$\mathcal{T}(q) = \lim_{t \rightarrow +\infty} \mathcal{T}(\gamma(t)) = \lim_{t \rightarrow +\infty} [\mathcal{T}(\gamma(0)) + c_\gamma t] = +\infty,$$

a contradiction.

Similarly one can show that γ does not have a past endpoint. \square

Proposition 9. Let \mathcal{M} be a globally hyperbolic spacetime and let $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{M} \rightarrow \mathbb{R}$ be Cauchy time functions. Then the map $\sim : C_{\mathcal{T}_1}^{\mathbb{R}} \rightarrow C_{\mathcal{T}_2}^{\mathbb{R}}$, defined via

$$\forall \gamma \in C_{\mathcal{T}_1}^{\mathbb{R}} \quad \tilde{\gamma} := \gamma \circ (\mathcal{T}_2 \circ \gamma)^{-1} \circ \mathcal{T}_1 \circ \gamma \tag{14}$$

is a bijection¹⁰.

⁹As a side remark, notice it also explains why one cannot replace ‘surjection’ with ‘bijection’ in the statement of Proposition 8.

¹⁰As a side note, let us remark that (14) might also be written in the form $\tilde{\gamma} := \mathcal{T}_2|_{[\gamma]}^{-1} \circ \mathcal{T}_1|_{[\gamma]} \circ \gamma$.

Proof First, observe that formula (14) produces well-defined elements of $C_{\mathcal{T}_2}^{\mathbb{R}}$. Indeed, since $\mathcal{T}_1, \mathcal{T}_2$ are Cauchy time functions and any $\gamma \in C_{\mathcal{T}_1}^{\mathbb{R}}$ is inextendible (Proposition 8), the maps $(\mathcal{T}_i \circ \gamma)$, $i = 1, 2$ are continuous strictly increasing bijections of \mathbb{R} onto itself and so are their inverses. Hence $\tilde{\gamma}$ is a reparametrization of γ and thus it is an inextendible causal curve. Moreover, observe that $\mathcal{T}_2 \circ \tilde{\gamma} = \mathcal{T}_1 \circ \gamma$ and therefore

$$\forall s, t \in \mathbb{R} \quad \mathcal{T}_2(\tilde{\gamma}(t)) - \mathcal{T}_2(\tilde{\gamma}(s)) = \mathcal{T}_1(\gamma(t)) - \mathcal{T}_1(\gamma(s)) = c_\gamma(t - s),$$

which completes the proof that $\tilde{\gamma} \in C_{\mathcal{T}_2}^{\mathbb{R}}$.

Having showed that $\gamma \mapsto \tilde{\gamma}$ is a well-defined map, we immediately see that it is a bijection, its inverse being the map

$$C_{\mathcal{T}_2}^{\mathbb{R}} \ni \rho \mapsto \rho \circ (\mathcal{T}_1 \circ \rho)^{-1} \circ \mathcal{T}_2 \circ \rho.$$

Notice that (quite expectedly) the formulas for \sim and its inverse differ only in that \mathcal{T}_1 and \mathcal{T}_2 are swapped, and so the inverse map is well defined by a completely analogous reasoning to the one conducted above. \square

Our goal is to show that \sim is actually a *homeomorphism*, provided $\mathcal{T}_1, \mathcal{T}_2$ are Cauchy *temporal* functions. To this end we shall need several technical lemmas, some of which, however, elucidate some further properties of the spaces $C_{\mathcal{T}}^I$.

We begin with the following simple fact from Riemannian geometry.

Lemma 1. *Let \mathcal{M} be a connected Riemannian manifold equipped with a complete Riemannian metric h . Then, any $F \in C^\infty(\mathcal{M})$ is locally Lipschitz continuous, i.e.*

$$\forall \text{ compact } \mathcal{K} \subseteq \mathcal{M} \exists L > 0 \forall p, q \in \mathcal{K} \quad |F(p) - F(q)| \leq L d(p, q)$$

Proof Fix the compact set \mathcal{K} , take any $p, q \in \mathcal{K}$ and let $\gamma : [0, 1] \rightarrow \mathcal{M}$ be the minimizing geodesic connecting p and q (existing by the Hopf–Rinow theorem). Defining the gradient $\text{grad}_h F$ via $h(\text{grad}_h F, \cdot) = dF$ and the magnitude $|v|_h := \sqrt{h(v, v)}$ for any $v \in T\mathcal{M}$, one has that

$$\begin{aligned} |F(p) - F(q)| &= \left| \int_0^1 (F \circ \gamma)'(\tau) d\tau \right| \leq \int_0^1 |h(\text{grad}_h F(\gamma(\tau)), \gamma'(\tau))| d\tau \\ &\leq \int_0^1 |\text{grad}_h F(\gamma(\tau))|_h \cdot |\gamma'(\tau)|_h d\tau \leq \max_{r \in \mathcal{K}} |\text{grad}_h F(r)|_h \cdot \int_0^1 |\gamma'(\tau)|_h d\tau \\ &= \max_{r \in \mathcal{K}} |\text{grad}_h F(r)|_h \cdot d(p, q), \end{aligned}$$

where we have used the Cauchy–Schwarz inequality. The maximum exists because, by the smoothness of F , the map $r \mapsto |\text{grad}_h F(r)|_h$ is continuous and hence it attains its extremes on \mathcal{K} . \square

Lemma 2. *Let \mathcal{M} be a stably causal spacetime, $\mathcal{T} : \mathcal{M} \rightarrow \mathbb{R}$ a time function and $I \subseteq \mathbb{R}$ an interval. Assume that $\gamma_n \rightarrow \gamma$ in $C_{\mathcal{T}}^I$. Then the positive sequence (c_{γ_n}) converges to $c_\gamma > 0$.*

Proof Fix any $a, b \in I$. Using (6), the continuity of \mathcal{T} and the fact that convergence in the compact-open topology implies pointwise convergence, one obtains

$$c_{\gamma_n} = \frac{\mathcal{T}(\gamma_n(b)) - \mathcal{T}(\gamma_n(a))}{b - a} \rightarrow \frac{\mathcal{T}(\gamma(b)) - \mathcal{T}(\gamma(a))}{b - a} = c_\gamma.$$

□

In the remaining lemmas, \mathcal{M} will always be a globally hyperbolic spacetime.

Lemma 3. *Let $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{M} \rightarrow \mathbb{R}$ be smooth time functions and let $I \subseteq \mathbb{R}$ be an interval. Assume that $\gamma_n \rightarrow \gamma$ in $C_{\mathcal{T}_1}^I$. Then $\mathcal{T}_2 \circ \gamma_n \rightarrow \mathcal{T}_2 \circ \gamma$ in $C(I, \mathbb{R})$ uniformly on compact sets.*

Proof It suffices to prove that $\mathcal{T}_2 \circ \gamma_n \rightarrow \mathcal{T}_2 \circ \gamma$ uniformly on any $[a, b] \subseteq I$. To begin with, observe that, since $\gamma_n|_{[a, b]} \rightarrow \gamma|_{[a, b]}$ uniformly, therefore the sequences $(\gamma_n(a)), (\gamma_n(b)) \subseteq \mathcal{M}$ are convergent and hence bounded. Define

$$\mathcal{K} := J^+ \left(\overline{\{\gamma_n(a) \mid n \in \mathbb{N}\}} \right) \cap J^- \left(\overline{\{\gamma_n(b) \mid n \in \mathbb{N}\}} \right), \quad (15)$$

which is a compact subset of \mathcal{M} on the strength of Proposition 2, *i*). What is more, \mathcal{K} contains the images $\gamma([a, b])$ and $\gamma_n([a, b])$ for all $n \in \mathbb{N}$. Therefore, on the strength of Lemma 1, there exists $L > 0$ such that

$$\sup_{t \in [a, b]} |\mathcal{T}_2(\gamma_n(t)) - \mathcal{T}_2(\gamma(t))| \leq L \sup_{t \in [a, b]} d(\gamma_n(t), \gamma(t)) \rightarrow 0.$$

□

Lemma 4. *Let $\mathcal{T} : \mathcal{M} \rightarrow \mathbb{R}$ be a Cauchy temporal function and let $I \subseteq \mathbb{R}$ be an interval. Assume that $\gamma_n \rightarrow \gamma$ in $C_{\mathcal{T}}^I$. Then*

$$\forall [a, b] \subseteq I \ \exists l_{a, b} > 0 \ \forall n \in \mathbb{N} \ \forall s, t \in [a, b] \quad \frac{1}{l_{a, b}} |s - t| \leq d_w(\gamma_n(s), \gamma_n(t)) \leq l_{a, b} |s - t|, \quad (16)$$

where d_w is the distance function associated to the complete Riemannian metric w introduced in Section 2.

Proof Fix $[a, b] \subseteq I$ and define the compact set \mathcal{K} by (15). The first inequality in (16) can be proven by noticing that for any $s, t \in [a, b]$ and any $n \in \mathbb{N}$

$$d_w(\gamma_n(s), \gamma_n(t)) \geq \frac{1}{L} |\mathcal{T}(\gamma_n(s)) - \mathcal{T}(\gamma_n(t))| = \frac{c_{\gamma_n}}{L} |s - t| \geq \underbrace{\frac{1}{L} \inf_{n \in \mathbb{N}} c_{\gamma_n}}_{=: l_1} |s - t|,$$

where the existence of the positive constant L is guaranteed by Lemma 1, $\inf_{n \in \mathbb{N}} c_{\gamma_n} > 0$ by Lemma 2 and the equality follows from (5).

We now move to the second inequality in (16). Fix $n \in \mathbb{N}$ and recall that $\gamma'_n(\tau)$ exists and is a future-directed causal vector for $\tau \in [a, b]$ a.e. [24, Remark 3.18], which means that

$$g(\gamma'_n(\tau), \gamma'_n(\tau)) \leq 0 \quad (17)$$

for $\tau \in [a, b]$ a.e. Notice, moreover, that for such τ (6) implies that

$$d\mathcal{T}(\gamma'_n(\tau)) = (\mathcal{T} \circ \gamma_n)'(\tau) = \lim_{\xi \rightarrow \tau} \frac{\mathcal{T}(\gamma_n(\xi)) - \mathcal{T}(\gamma_n(\tau))}{\xi - \tau} = c_{\gamma_n}. \quad (18)$$

Assuming $s < t$ and using (4), (17) and (18), one obtains

$$\begin{aligned} d_w(\gamma_n(s), \gamma_n(t)) &\leq \int_s^t \sqrt{w(\gamma'_n(\tau), \gamma'_n(\tau))} d\tau \\ &= \int_s^t \sqrt{u(\gamma_n(\tau))} \cdot \sqrt{g(\gamma'_n(\tau), \gamma'_n(\tau)) + 2\alpha(\gamma_n(\tau)) [d\mathcal{T}(\gamma'_n(\tau))]^2} d\tau \\ &\leq c_{\gamma_n} \int_s^t \sqrt{2u(\gamma_n(\tau)) \cdot \alpha(\gamma_n(\tau))} d\tau \leq \underbrace{\sup_{n \in \mathbb{N}} c_{\gamma_n} \cdot \max_{p \in \mathcal{K}} \sqrt{2u(p)\alpha(p)}}_{=: l_2} \cdot |s - t|, \end{aligned}$$

where the compactness of \mathcal{K} assures that the continuous map $p \mapsto \sqrt{2u(p)\alpha(p)}$ attains its maximum on \mathcal{K} . The sequence (c_{γ_n}) is bounded by Lemma 2.

To finish the proof of (16), we obviously take $l_{a,b} := \max\{1/l_1, l_2\}$. \square

Property (16) could be called the *local bi-Lipschitz equicontinuity* of $\{\gamma_n\}_{n \in \mathbb{N}} \subseteq C_{\mathcal{T}}^I$ (compare [4, pp. 75–76]). The next lemma shows that this property remains valid if we compose γ_n 's with another Cauchy temporal function.

Lemma 5. *Let $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{M} \rightarrow \mathbb{R}$ be Cauchy temporal functions and let $I \subseteq \mathbb{R}$ be an interval. Assume that $\gamma_n \rightarrow \gamma$ in $C_{\mathcal{T}_1}^I$. Then*

$$\begin{aligned} \forall [a, b] \subseteq I \exists L_{a,b} > 0 \forall n \in \mathbb{N} \forall s, t \in [a, b] \\ \frac{1}{L_{a,b}} |s - t| \leq |\mathcal{T}_2(\gamma_n(s)) - \mathcal{T}_2(\gamma_n(t))| \leq L_{a,b} |s - t|. \end{aligned} \quad (19)$$

Proof Fix $[a, b] \subseteq I$. For any fixed $n \in \mathbb{N}$, by (2), (17) and (18) we know that

$$\begin{aligned} 0 &\geq g(\gamma'_n(\tau), \gamma'_n(\tau)) = -\alpha(\gamma_n(\tau)) [d\mathcal{T}_1(\gamma'_n(\tau))]^2 + \bar{g}(\gamma'_n(\tau), \gamma'_n(\tau)) \\ &= -\alpha(\gamma_n(\tau)) c_{\gamma_n}^2 + \bar{g}(\gamma'_n(\tau), \gamma'_n(\tau)) \end{aligned}$$

for $\tau \in [a, b]$ a.e. and hence, for such τ ,

$$\sqrt{\bar{g}(\gamma'_n(\tau), \gamma'_n(\tau))} \leq c_{\gamma_n} \sqrt{\alpha(\gamma_n(\tau))}. \quad (20)$$

Furthermore, (2) gives us that

$$[d\mathcal{T}_1(\text{grad } \mathcal{T}_2)]^2 = \frac{\bar{g}(\text{grad } \mathcal{T}_2, \text{grad } \mathcal{T}_2) - g(\text{grad } \mathcal{T}_2, \text{grad } \mathcal{T}_2)}{\alpha}. \quad (21)$$

Notice that $d\mathcal{T}_1(\text{grad } \mathcal{T}_2) = (\text{grad } \mathcal{T}_2)(\mathcal{T}_1) < 0$, because $\mathcal{T}_1, \mathcal{T}_2$ are temporal functions. Therefore, taking the square root of (21) yields

$$d\mathcal{T}_1(\text{grad } \mathcal{T}_2) = -\frac{1}{\sqrt{\alpha}} \sqrt{\bar{g}(\text{grad } \mathcal{T}_2, \text{grad } \mathcal{T}_2) - g(\text{grad } \mathcal{T}_2, \text{grad } \mathcal{T}_2)}. \quad (22)$$

Altogether, for any $\tau \in [a, b]$ at which γ'_n exists we obtain that

$$\begin{aligned}
(\mathcal{T}_2 \circ \gamma_n)'(\tau) &= g(\text{grad } \mathcal{T}_2, \gamma'_n) = -\alpha d\mathcal{T}_1(\text{grad } \mathcal{T}_2) d\mathcal{T}_1(\gamma'_n) + \bar{g}(\text{grad } \mathcal{T}_2, \gamma'_n) \\
&\geq -c_{\gamma_n} \alpha d\mathcal{T}_1(\text{grad } \mathcal{T}_2) - \sqrt{\bar{g}(\text{grad } \mathcal{T}_2, \text{grad } \mathcal{T}_2)} \sqrt{\bar{g}(\gamma'_n, \gamma'_n)} \\
&\geq c_{\gamma_n} \underbrace{\sqrt{\alpha} \left[\sqrt{\bar{g}(\text{grad } \mathcal{T}_2, \text{grad } \mathcal{T}_2)} - g(\text{grad } \mathcal{T}_2, \text{grad } \mathcal{T}_2) - \sqrt{\bar{g}(\text{grad } \mathcal{T}_2, \text{grad } \mathcal{T}_2)} \right]}_{=: G},
\end{aligned} \tag{23}$$

where we have suppressed arguments of functions and then used (2), (18), the Cauchy-Schwarz inequality for \bar{g} , (20) and (22). Notice that the newly defined function $G : \mathcal{M} \rightarrow \mathbb{R}$ is continuous and positive, because \mathcal{T}_2 is temporal and hence $-g(\text{grad } \mathcal{T}_2, \text{grad } \mathcal{T}_2) > 0$.

Assuming $s < t$, with the help of (23) we can now obtain the first inequality in (19). Namely,

$$\mathcal{T}_2(\gamma_n(t)) - \mathcal{T}_2(\gamma_n(s)) = \int_s^t (\mathcal{T}_2 \circ \gamma_n)'(\tau) d\tau \geq \underbrace{\inf_{n \in \mathbb{N}} c_{\gamma_n} \cdot \min_{r \in \mathcal{K}} G(r)}_{=: L_1} \cdot (t - s),$$

where \mathcal{K} is the compact set defined by (15) and thus G attains on \mathcal{K} its minimum, which is positive. Lemma 4 assures the positivity of $\inf_n c_{\gamma_n}$.

As for the second inequality in (19), it is a straightforward consequence of Lemmas 1 and 4, on the strength of which one has

$$\forall n \in \mathbb{N} \forall s, t \in [a, b] \quad |\mathcal{T}_2(\gamma_n(s)) - \mathcal{T}_2(\gamma_n(t))| \leq l_{A,B} d_w(\gamma_n(s), \gamma_n(t)) \leq \underbrace{l_{A,B} L}_{=: L_2} |s - t|,$$

where $A := \min_{r \in \mathcal{K}} \mathcal{T}_2(r)$ and $B := \max_{r \in \mathcal{K}} \mathcal{T}_2(r)$.

To finish the proof of (19), we obviously take $L_{a,b} := \max\{1/L_1, L_2\}$. \square

Lemma 6. *Under the assumptions of Lemma 5 with $I := \mathbb{R}$ it is true that*

$$(\mathcal{T}_2 \circ \gamma_n)^{-1} \circ \mathcal{T}_1 \circ \gamma_n \rightarrow (\mathcal{T}_2 \circ \gamma)^{-1} \circ \mathcal{T}_1 \circ \gamma$$

in $C(\mathbb{R}, \mathbb{R})$ uniformly on compact sets.

Proof It suffices to show that $(\mathcal{T}_2 \circ \gamma_n)^{-1} \circ \mathcal{T}_1 \circ \gamma_n \rightarrow (\mathcal{T}_2 \circ \gamma)^{-1} \circ \mathcal{T}_1 \circ \gamma$ uniformly on any $[a, b] \subseteq \mathbb{R}$. For any $t \in [a, b]$ one has

$$\begin{aligned}
&|(\mathcal{T}_2 \circ \gamma_n)^{-1}(\mathcal{T}_1(\gamma_n(t))) - (\mathcal{T}_2 \circ \gamma)^{-1}(\mathcal{T}_1(\gamma(t)))| \\
&\leq |(\mathcal{T}_2 \circ \gamma_n)^{-1}(\mathcal{T}_1(\gamma_n(t))) - (\mathcal{T}_2 \circ \gamma_n)^{-1}(\mathcal{T}_1(\gamma(t)))| \\
&\quad + |(\mathcal{T}_2 \circ \gamma_n)^{-1}(\mathcal{T}_1(\gamma(t))) - (\mathcal{T}_2 \circ \gamma)^{-1}(\mathcal{T}_1(\gamma(t)))|
\end{aligned} \tag{24}$$

Both terms on the right-hand side can be estimated from above with the help of Lemma 5 (first inequality). For the first term one has

$$|(\mathcal{T}_2 \circ \gamma_n)^{-1}(\mathcal{T}_1(\gamma_n(t))) - (\mathcal{T}_2 \circ \gamma_n)^{-1}(\mathcal{T}_1(\gamma(t)))| \leq L_{A,B} |\mathcal{T}_1(\gamma_n(t)) - \mathcal{T}_1(\gamma(t))|,$$

where $A := \min_{r \in \mathcal{K}} \mathcal{T}_1(r)$ and $B := \max_{r \in \mathcal{K}} \mathcal{T}_1(r)$, in which \mathcal{K} is again the compact set given by (15).

To estimate the second term, it is convenient to introduce the auxiliary variable $\tau := (\mathcal{T}_2 \circ \gamma)^{-1}(\mathcal{T}_1(\gamma(t)))$. Observe that $\tau \in [a', b']$, where $a' := (\mathcal{T}_2 \circ \gamma)^{-1}(\mathcal{T}_1(\gamma(a)))$ and $b' := (\mathcal{T}_2 \circ \gamma)^{-1}(\mathcal{T}_1(\gamma(b)))$, because the map $(\mathcal{T}_2 \circ \gamma)^{-1} \circ \mathcal{T}_1 \circ \gamma$ is continuous and strictly increasing. One can now write that

$$\begin{aligned} & |(\mathcal{T}_2 \circ \gamma_n)^{-1}(\mathcal{T}_1(\gamma(t))) - (\mathcal{T}_2 \circ \gamma)^{-1}(\mathcal{T}_1(\gamma(t)))| = |(\mathcal{T}_2 \circ \gamma_n)^{-1}(\mathcal{T}_2(\gamma(\tau))) - \tau| \\ & = |(\mathcal{T}_2 \circ \gamma_n)^{-1}(\mathcal{T}_2(\gamma(\tau))) - (\mathcal{T}_2 \circ \gamma_n)^{-1}(\mathcal{T}_2(\gamma_n(\tau)))| \leq L_{A', B'} |\mathcal{T}_2(\gamma(\tau)) - \mathcal{T}_2(\gamma_n(\tau))|, \end{aligned}$$

where $A' := \min_{r \in \mathcal{K}'} \mathcal{T}_2(r)$ and $B' := \max_{r \in \mathcal{K}'} \mathcal{T}_2(r)$, in which \mathcal{K}' is defined as

$$\mathcal{K}' := J^+ \left(\overline{\{\gamma_n(a') \mid n \in \mathbb{N}\}} \right) \cap J^- \left(\overline{\{\gamma_n(b') \mid n \in \mathbb{N}\}} \right),$$

i.e. it is another compact subset of \mathcal{M} designed so as to contain the images $\gamma([a', b'])$ and $\gamma_n([a', b'])$ for all $n \in \mathbb{N}$.

Applying the above estimates to (24) and taking the supremum over $t \in [a, b]$, one obtains that

$$\begin{aligned} & \sup_{t \in [a, b]} |(\mathcal{T}_2 \circ \gamma_n)^{-1}(\mathcal{T}_1(\gamma_n(t))) - (\mathcal{T}_2 \circ \gamma)^{-1}(\mathcal{T}_1(\gamma(t)))| \\ & \leq L_{A, B} \sup_{t \in [a, b]} |\mathcal{T}_1(\gamma_n(t)) - \mathcal{T}_1(\gamma(t))| + L_{A', B'} \sup_{\tau \in [a', b']} |\mathcal{T}_2(\gamma(\tau)) - \mathcal{T}_2(\gamma_n(\tau))|, \end{aligned}$$

which tends to zero on the strength of Lemma 3. \square

We are finally ready to strengthen Proposition 9.

Theorem 3. *Let $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{M} \rightarrow \mathbb{R}$ be Cauchy temporal functions. Then the map $\tilde{\cdot} : C_{\mathcal{T}_1}^{\mathbb{R}} \rightarrow C_{\mathcal{T}_2}^{\mathbb{R}}$, defined via (14) is a homeomorphism.*

Proof On the strength of Proposition 9, we only need to show that $\gamma_n \rightarrow \gamma$ in $C_{\mathcal{T}_1}^{\mathbb{R}}$ iff $\tilde{\gamma}_n \rightarrow \tilde{\gamma}$ in $C_{\mathcal{T}_2}^{\mathbb{R}}$. In fact, showing just one of the implications is enough, because proving the other one amounts to swapping \mathcal{T}_1 with \mathcal{T}_2 .

Thus, assume that $\gamma_n \rightarrow \gamma$ in $C_{\mathcal{T}_1}^{\mathbb{R}}$ and for any $n \in \mathbb{N}$ denote $\lambda_n := (\mathcal{T}_2 \circ \gamma_n)^{-1} \circ \mathcal{T}_1 \circ \gamma_n$ as well as $\lambda := (\mathcal{T}_2 \circ \gamma)^{-1} \circ \mathcal{T}_1 \circ \gamma$. Our goal is to show that $\tilde{\gamma}_n = \gamma_n \circ \lambda_n \rightarrow \gamma \circ \lambda = \tilde{\gamma}$ uniformly on any $[a, b] \subseteq \mathbb{R}$, as this already implies convergence in $C_{\mathcal{T}_2}^{\mathbb{R}}$.

The sequence $(\lambda_n) \subseteq C(\mathbb{R}, \mathbb{R})$ consists of strictly increasing maps, which by Lemma 6 converges to the strictly increasing map $\lambda \in C(\mathbb{R}, \mathbb{R})$ uniformly on compact sets. The latter implies the pointwise convergence, therefore the sequences $(\lambda_n(a)), (\lambda_n(b)) \subseteq \mathcal{M}$ are convergent and hence bounded. Denote $A := \inf_n \lambda_n(a)$ and $B := \sup_n \lambda_n(b)$. Notice that, because λ and λ_n for every $n \in \mathbb{N}$ are continuous and strictly increasing maps, the interval $[A, B]$ contains $\lambda([a, b]) = [\lambda(a), \lambda(b)]$ as well as $\lambda_n([a, b]) = [\lambda_n(a), \lambda_n(b)]$ for all $n \in \mathbb{N}$.

For any $t \in [a, b]$ one obtains

$$\begin{aligned} d_w(\gamma_n(\lambda_n(t)), \gamma(\lambda(t))) & \leq d_w(\gamma_n(\lambda_n(t)), \gamma_n(\lambda(t))) + d_w(\gamma_n(\lambda(t)), \gamma(\lambda(t))) \\ & \leq L_{A, B} |\lambda_n(t) - \lambda(t)| + d_w(\gamma_n(\lambda(t)), \gamma(\lambda(t))) \end{aligned}$$

where $L_{A, B} > 0$ exists on the strength of Lemma 4. Taking the supremum over $t \in [a, b]$, one obtains

$$\sup_{t \in [a, b]} d_w(\gamma_n(\lambda_n(t)), \gamma(\lambda(t))) \leq L_{A, B} \sup_{t \in [a, b]} |\lambda_n(t) - \lambda(t)| + \sup_{\tau \in [\lambda(a), \lambda(b)]} d_w(\gamma_n(\tau), \gamma(\tau)).$$

Notice now that both terms on the right-hand side tend to zero. Indeed, the rightmost term does so by assumption, whereas the other one by Lemma 6. \square

4 Application in Lorentzian optimal transport theory

Section 2.1 began with recalling one of the fundamentals of causality theory – the causal precedence relation \preceq between points of a given spacetime \mathcal{M} . In [13] we have proposed how to extend this relation onto $\mathcal{P}(\mathcal{M})$.

Definition 3. *Let \mathcal{M} be a spacetime. For any $\mu, \nu \in \mathcal{P}(\mathcal{M})$ we say that μ causally precedes ν (symbolically $\mu \preceq \nu$) iff there exists $\omega \in \mathcal{P}(\mathcal{M}^2)$ such that*

$$i) \pi_{\#}^1 \omega = \mu \text{ and } \pi_{\#}^2 \omega = \nu,$$

$$ii) \omega(J^+) = 1,$$

Let us emphasize that the left-hand side of the latter condition is well-defined, because J^+ is a Borel subset of \mathcal{M}^2 [13, Section 3].

In the optimal transport theory, any $\omega \in \mathcal{P}(\mathcal{M}^2)$ satisfying *i)* from the above definition is called a *coupling* of μ and ν [32]. If it satisfies both conditions, it is called a *causal coupling*¹¹ of μ and ν . We shall write $\Pi(\mu, \nu)$ ($\Pi_c(\mu, \nu)$) for the set of all (causal) couplings of μ and ν .

The relation \preceq is reflexive and transitive. If \mathcal{M} is (past or future) distinguishing, then \preceq is also antisymmetric and hence a partial order [13]. In any case, its irreflexive kernel will be denoted by \prec .

For spacetimes with sufficiently good causal properties, the following characterization of \preceq holds [13].

Theorem 4. *Let \mathcal{M} be a causally simple spacetime and let $\mu, \nu \in \mathcal{P}(\mathcal{M})$. Then $\mu \preceq \nu$ iff $\mu(J^+(\mathcal{K})) \leq \nu(J^+(\mathcal{K}))$ for any compact $\mathcal{K} \subseteq \mathcal{M}$.*

The Polish spaces of causal curves introduced in the previous section allow to further develop the said extension of causality theory. The main result of this chapter, stated below, might be regarded as a (loose) analogue of [2, Theorem 2.10], in which geodesics in a Polish geodesic space are replaced with causal curves in a globally hyperbolic spacetime, or as a (distant) cousin of the result called the “superposition principle” as given in [7, Theorem 3] (see also [1, Theorem 3.2] or [12, Theorem 6.2.2] for other formulations).

In the present chapter, we focus on the purely mathematical side of the said result. The discussion of its physical content is carried out in Section 5.

Theorem 5. *Let \mathcal{M} be a globally hyperbolic spacetime and let $\mathcal{T} : \mathcal{M} \rightarrow \mathbb{R}$ be a Cauchy temporal function. Let also $I \subseteq \mathbb{R}$ be any interval. Consider the map $\mu : I \rightarrow \mathcal{P}(\mathcal{M})$, $t \mapsto \mu_t$ such that $\text{supp } \mu_t \subseteq \mathcal{T}^{-1}(t)$ for every $t \in I$. The following conditions are equivalent*

i) The map μ is causal in the sense that

$$\forall s, t \in I \quad s \leq t \Rightarrow \mu_s \preceq \mu_t, \tag{25}$$

ii) There exists $\sigma \in \mathcal{P}(C_{\mathcal{T}}^I)$ such that $(\text{ev}_t)_{\#} \sigma = \mu_t$ for every $t \in I$.

Remark 4. Condition (25) is analogous to the characterization of causal curves given in Proposition 1. Indeed, since we assume that $\text{supp } \mu_t \subseteq \mathcal{T}^{-1}(t)$ for all $t \in I$, therefore it is not possible that for some distinct $s, t \in I$ one has $\mu_s = \mu_t$. Summarizing, (25) is equivalent to $\forall s, t \in I \quad s < t \Rightarrow \mu_s \prec \mu_t$.

¹¹The name was independently coined in [31].

Observe, however, that causal curves are implicitly *continuous*. Therefore, it seems that we should, by analogy, impose the narrow continuity of the map $t \mapsto \mu_t$. However, we will show below (Proposition 11) that the narrow continuity is already guaranteed by condition *i*) and so there is no need to assume it separately.

Proof of Theorem 5, *ii*) \Rightarrow *i*). Let $\iota : J^+ \hookrightarrow \mathcal{M}^2$ denote the canonical topological embedding and take any $s, t \in I$ such that $s \leq t$. The map $\text{ev}_s \times \text{ev}_t : C_{\mathcal{T}}^I \rightarrow J^+$, $\gamma \mapsto (\gamma(s), \gamma(t))$ is well-defined and continuous, hence Borel. One can thus define $\omega_{s,t} := [\iota \circ (\text{ev}_s \times \text{ev}_t)]_{\#} \sigma$. We show that $\omega_{s,t} \in \Pi_c(\mu_s, \mu_t)$.

Indeed, $\pi_{\#}^1 \omega_{s,t} = [\pi^1 \circ \iota \circ (\text{ev}_s \times \text{ev}_t)]_{\#} \sigma = (\text{ev}_s)_{\#} \sigma = \mu_s$ and similarly $\pi_{\#}^2 \omega_{s,t} = \mu_t$. Additionally,

$$\omega_{s,t}(J^+) = \sigma((\text{ev}_s \times \text{ev}_t)^{-1}(J^+)) = \sigma(C_{\mathcal{T}}^I) = 1,$$

which completes the proof of *i*). \square

The proof of the converse implication requires some technical preparations. To begin with, we provide some auxiliary properties of the narrow topology.

Lemma 7. *Let \mathcal{X}, \mathcal{Y} be Polish spaces and let $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a continuous map. Then $F_{\#} : \mathcal{P}(\mathcal{X}) \rightarrow \mathcal{P}(\mathcal{Y})$ is continuous (in the narrow topology). Moreover, if F is also proper, then so is $F_{\#}$.*

Proof Suppose that $(\mu_n) \subseteq \mathcal{P}(\mathcal{X})$ is narrowly convergent to $\mu \in \mathcal{P}(\mathcal{X})$. Take $g \in C_b(\mathcal{Y})$. One obtains

$$\int_{\mathcal{Y}} g d(F_{\#} \mu_n) = \int_{\mathcal{X}} \underbrace{(g \circ F)}_{\in C_b(\mathcal{X})} d\mu_n \rightarrow \int_{\mathcal{X}} (g \circ F) d\mu = \int_{\mathcal{Y}} g d(F_{\#} \mu),$$

what proves that $F_{\#}$ is continuous.

Assume now that F is continuous and proper. Let $\mathcal{K} \subseteq \mathcal{P}(\mathcal{Y})$ be compact. We want to show that $F_{\#}^{-1}(\mathcal{K})$ is a compact subset of $\mathcal{P}(\mathcal{X})$. By the continuity of $F_{\#}$, $F_{\#}^{-1}(\mathcal{K})$ is closed, and so it suffices to additionally show that it is relatively compact. By the Prokhorov theorem, this is, in turn, equivalent to showing that $F_{\#}^{-1}(\mathcal{K})$ is *tight*, i.e.

$$\forall \varepsilon > 0 \quad \exists \text{ compact } K_{\varepsilon} \subseteq \mathcal{X} \quad \forall \mu \in F_{\#}^{-1}(\mathcal{K}) \quad \mu(K_{\varepsilon}) \geq 1 - \varepsilon.$$

Let us thus fix $\varepsilon > 0$. By assumption, $\mathcal{K} \subseteq \mathcal{P}(\mathcal{Y})$ is compact and hence tight, therefore there exists $K'_{\varepsilon} \subseteq \mathcal{Y}$ compact and such that $\nu(K'_{\varepsilon}) \geq 1 - \varepsilon$ for every $\nu \in \mathcal{K}$. In particular, this is true for $\nu = F_{\#} \mu$ for every $\mu \in F_{\#}^{-1}(\mathcal{K})$. We thus obtain that

$$\forall \mu \in F_{\#}^{-1}(\mathcal{K}) \quad \mu(F^{-1}(K'_{\varepsilon})) \geq 1 - \varepsilon.$$

By the assumption that F is proper, taking $K_{\varepsilon} := F^{-1}(K'_{\varepsilon})$ completes the proof. \square

Lemma 8. *Let \mathcal{X} be a second-countable LCH space and suppose that the family of measures $\{\mu_n\} \subseteq \mathcal{P}(\mathcal{X})$ is tight. Then (μ_n) converges to $\mu \in \mathcal{P}(\mathcal{X})$ narrowly iff $\forall f \in C_c(\mathcal{X}) \int_{\mathcal{X}} f d\mu_n \rightarrow \int_{\mathcal{X}} f d\mu$.*

Proof The “ \Rightarrow ” part is trivial. To prove the “ \Leftarrow ” part, recall that every second countable LCH space \mathcal{X} admits an *exhaustion by compact sets*, i.e. a sequence (\mathcal{K}_m) of compact subsets of \mathcal{X} such that $\mathcal{K}_m \subseteq \text{int } \mathcal{K}_{m+1}$ for all m and $\bigcup_{m \in \mathbb{N}} \mathcal{K}_m = \mathcal{X}$. Furthermore, let $(\varphi_m) \subseteq C_c(\mathcal{X})$ be a sequence of functions satisfying $0 \leq \varphi_m \leq 1$, $\varphi_m|_{\mathcal{K}_m} \equiv 1$ and $\text{supp } \varphi_m \subseteq \mathcal{K}_{m+1}$, existing by Urysohn’s lemma. Take any $f \in C_b(\mathcal{X})$. Then, by assumption,

$$\forall m \in \mathbb{N} \quad \lim_{n \rightarrow +\infty} \int_{\mathcal{X}} f \varphi_m d\mu_n = \int_{\mathcal{X}} f \varphi_m d\mu.$$

Since \mathcal{X} is Polish, μ is inner regular, which, together with the tightness of $\{\mu_n\}$, means that

$$\forall \varepsilon > 0 \exists \text{ compact } K \subseteq \mathcal{X} \quad \mu(K^c) \leq \varepsilon \quad \text{and} \quad \forall n \in \mathbb{N} \quad \mu_n(K^c) \leq \varepsilon.$$

We claim that $\int_{\mathcal{X}} f d\mu_n \rightarrow \int_{\mathcal{X}} f d\mu$. Indeed, fix $\varepsilon > 0$ and take $K \subseteq \mathcal{X}$ compact and such that

$$\mu(K^c) \leq \frac{\varepsilon}{4\|f\|} \quad \text{and} \quad \forall n \in \mathbb{N} \quad \mu_n(K^c) \leq \frac{\varepsilon}{4\|f\|},$$

where $\|f\|$ denotes the supremum of f . We now have that

$$\begin{aligned} \left| \int_{\mathcal{X}} f d\mu_n - \int_{\mathcal{X}} f d\mu \right| &= \left| \int_{\mathcal{X}} f \varphi_m d\mu_n - \int_{\mathcal{X}} f \varphi_m d\mu + \int_{\mathcal{K}_m^c} f(1 - \varphi_m) d\mu_n - \int_{\mathcal{K}_m^c} f(1 - \varphi_m) d\mu \right| \\ &\leq \left| \int_{\mathcal{X}} f \varphi_m d\mu_n - \int_{\mathcal{X}} f \varphi_m d\mu \right| + \|f\| \mu_n(\mathcal{K}_m^c) + \|f\| \mu(\mathcal{K}_m^c). \end{aligned}$$

It now remains to take m large enough to have the inclusion $K \subseteq \mathcal{K}_m$ (which makes the two rightmost summands less than $\varepsilon/4$ each) and then choose $N \in \mathbb{N}$ such that the leftmost summand falls below $\varepsilon/2$ for all $n > N$. \square

Lemma 9. *Let \mathcal{M} be a causally simple spacetime and let $\mu, \nu \in \mathcal{P}(\mathcal{M})$. Then $\Pi_c(\mu, \nu)$ is a narrowly compact subset of $\mathcal{P}(\mathcal{M}^2)$.*

Proof It is well known that $\Pi(\mu, \nu)$ is narrowly compact in $\mathcal{P}(\mathcal{M}^2)$ [2, Theorem 1.5]. Hence we only need to show that $\Pi_c(\mu, \nu)$ is closed in $\Pi(\mu, \nu)$. To this end, take any sequence $(\omega_n) \subseteq \Pi_c(\mu, \nu)$ convergent to some $\omega \in \Pi(\mu, \nu)$. We need to prove that $\omega(J^+) = 1$. This, however, is a direct consequence of the portmanteau theorem, on the strength of which

$$\omega_n \rightarrow \omega \text{ narrowly} \Leftrightarrow \forall \text{ closed } C \subseteq \mathcal{M}^2 \quad \limsup_{n \rightarrow +\infty} \omega_n(C) \leq \omega(C).$$

Taking $C = J^+$ — which is closed by the causal simplicity of \mathcal{M} — we easily obtain

$$1 = \limsup_{n \rightarrow +\infty} \omega_n(J^+) \leq \omega(J^+) \leq 1,$$

and so $\omega(J^+) = 1$. \square

The next Proposition is an analogue of [31, Proposition 3.3].

Proposition 10. Let \mathcal{M} be a globally hyperbolic spacetime and let $\mathcal{T} : \mathcal{M} \rightarrow \mathbb{R}$ be a time function. For any $a, b \in \mathbb{R}$ the map $\text{ev}_a \times \text{ev}_b : C_{\mathcal{T}}^{[a,b]} \rightarrow J^+$ is proper and admits a Borel right inverse.

Proof To prove properness, let $\mathcal{K} \subseteq J^+$ be compact and observe that

$$(\text{ev}_a \times \text{ev}_b)^{-1}(\mathcal{K}) \subseteq (\text{ev}_a \times \text{ev}_b)^{-1}(\pi^1(\mathcal{K}) \times \pi^2(\mathcal{K}) \cap J^+) = C_{\mathcal{T}}^{[a,b]}(\pi^1(\mathcal{K}), \pi^2(\mathcal{K})),$$

which, by the continuity of the map $\text{ev}_a \times \text{ev}_b$ and by Proposition 5, means that $(\text{ev}_a \times \text{ev}_b)^{-1}(\mathcal{K})$ is a closed subset of a compact set and hence it is itself compact.

For the second part of the proposition's statement we use the standard measurable selection theorem, by which any surjective Borel map $F : \mathcal{X} \rightarrow \mathcal{Y}$ between Polish spaces such that $F^{-1}(y)$ is compact for any $y \in \mathcal{Y}$, admits a Borel right inverse [32, p. 104].

We already know that $C_{\mathcal{T}}^{[a,b]}$ is a Polish space (Proposition 6) and the same concerns J^+ , which is a closed subset of \mathcal{M}^2 for \mathcal{M} globally hyperbolic. The map $\text{ev}_a \times \text{ev}_b$ is continuous and hence Borel. By the very definition of the causal precedence relation, for any $(p, q) \in J^+$ there exists a causal curve connecting them, which by Proposition 4 can be reparametrized so that it becomes an element of $C_{\mathcal{T}}^{[a,b]}$. Finally, the compactness of $(\text{ev}_a \times \text{ev}_b)^{-1}(p, q) = C_{\mathcal{T}}^{[a,b]}(p, q)$ is guaranteed by Proposition 5. \square

Given a pair of continuous (causal) curves $\gamma_1 : [a, b] \rightarrow \mathcal{M}$, $\gamma_2 : [b, c] \rightarrow \mathcal{M}$ such that $\gamma_1(b) = \gamma_2(b)$, one can easily concatenate them, obtaining another continuous (causal) curve $\gamma_1 \sqcup \gamma_2 : [a, c] \rightarrow \mathcal{M}$ through the obvious piecewise definition. In the proof of Theorem 5, however, we will need a way to “concatenate” two *measures* on spaces of curves, one on $C_{\mathcal{T}}^{[a,b]}$ and the other on $C_{\mathcal{T}}^{[b,c]}$. To this end, recall first the *disintegration theorem* [3, Theorem 5.3.1].

Theorem 6. Let \mathcal{X}, \mathcal{Y} be Polish spaces, $\mu \in \mathcal{P}(\mathcal{Y})$ and let $\pi : \mathcal{Y} \rightarrow \mathcal{X}$ be a Borel map. Denote $\nu := \pi_{\#}\mu$. Then there exists a ν -a.e. uniquely determined family of probability measures $\{\mu^x\}_{x \in \mathcal{X}} \subseteq \mathcal{P}(\mathcal{Y})$ such that

- For any $E \in \mathcal{B}(\mathcal{Y})$ the map $\mathcal{X} \ni x \mapsto \mu^x(E)$ is Borel.
- Measures μ^x live on the fibers of π , that is $\mu^x(\mathcal{Y} \setminus \pi^{-1}(x)) = 0$ for $x \in \mathcal{X}$ ν -a.e.
- For any Borel map $f : \mathcal{Y} \rightarrow [0, +\infty]$

$$\int_{\mathcal{Y}} f d\mu = \int_{\mathcal{X}} \left(\int_{\pi^{-1}(x)} f(y) d\mu^x(y) \right) d\nu(x). \quad (26)$$

We call the family $\{\mu^x\}_{x \in \mathcal{X}}$ the *disintegration of μ with respect to (w.r.t.) π* .

Definition 4. Let \mathcal{M} be a stably causal spacetime and let $\mathcal{T} : \mathcal{M} \rightarrow \mathbb{R}$ be a time function. For any fixed $a, b, c \in \mathbb{R}$, $a < b < c$ let $\mathcal{Y} := \{(\gamma_1, \gamma_2) \in C_{\mathcal{T}}^{[a,b]} \times C_{\mathcal{T}}^{[b,c]} \mid \gamma_1(b) = \gamma_2(b)\}$ denote the (Polish) space of concatenable pairs of curves. Consider the concatenation map $\sqcup : \mathcal{Y} \rightarrow C_{\mathcal{T}}^{[a,c]}$, which is obviously continuous and hence Borel. For any $\sigma_1 \in \mathcal{P}(C_{\mathcal{T}}^{[a,b]})$ and $\sigma_2 \in \mathcal{P}(C_{\mathcal{T}}^{[b,c]})$, which are compatible in the sense that $(\text{ev}_b)_{\#}\sigma_1 = (\text{ev}_b)_{\#}\sigma_2 =: \nu$, we define their concatenation $\sigma_1 \sqcup \sigma_2 \in \mathcal{P}(C_{\mathcal{T}}^{[a,c]})$ via¹²

¹²That this is a good definition of a measure follows from the Riesz–Markov–Kakutani theorem. In short, one writes: $\sigma_1 \sqcup \sigma_2 := \int_{\mathcal{M}} \sqcup_{\#}(\sigma_1^x \times \sigma_2^x) d\nu(x)$. In order to convince oneself that the integral

$$\int_{C_{\mathcal{T}}^{[a,c]}} F d(\sigma_1 \sqcup \sigma_2) := \int_{\mathcal{M}} \left(\int_{\mathcal{Y}} F(\gamma_1 \sqcup \gamma_2) d(\sigma_1^x \times \sigma_2^x)(\gamma_1, \gamma_2) \right) d\nu(x), \quad (27)$$

for any $F \in C_c(C_{\mathcal{T}}^{[a,c]})$, where $\{\sigma_i^x\}_{x \in \mathcal{M}}$ is the disintegration of σ_i w.r.t. ev_b for $i = 1, 2$.

Remark 5. In the discussion preceding the disintegration theorem, as well as in Definition 4 everything is still valid if we replace $[a, b]$ with $(a, b]$ or even with $(-\infty, b]$ or if we replace $[b, c]$ with $[b, c)$ or even with $[b, +\infty)$. In other words, nothing prevents from concatenating measures living on the spaces of noncompact curves.

Remark 6. As one would expect, it is true (and easy to check) that

$$(\text{ev}_t)_{\#}(\sigma_1 \sqcup \sigma_2) = \begin{cases} (\text{ev}_t)_{\#}\sigma_1 & \text{for } t < b \\ \nu & \text{for } t = b \\ (\text{ev}_t)_{\#}\sigma_2 & \text{for } t > b \end{cases}.$$

In fact, this is the reason why we have chosen the symbol \sqcup to denote the concatenation operation in the first place.

We now prove the narrow continuity of a measure-valued map which is *causal* in the sense described in Theorem 5.

Proposition 11. Let \mathcal{M} be a globally hyperbolic spacetime and let $\mathcal{T} : \mathcal{M} \rightarrow \mathbb{R}$ be a Cauchy temporal function. Consider a map $\mu : I \rightarrow \mathcal{P}(\mathcal{M})$, $t \mapsto \mu_t$ such that $\text{supp } \mu_t \subseteq \mathcal{T}^{-1}(t)$ for every $t \in I$. If the map μ satisfies

$$\forall s, t \in I \quad s \leq t \Rightarrow \mu_s \preceq \mu_t, \quad (28)$$

then it is narrowly continuous.

Proof Fix any $a, b \in I$ with $a < b$. First, let us show that the family $\{\mu_t\}_{t \in [a,b]}$ is tight. By Lemma 8, this will allow us to use only the compactly supported test functions when proving the narrow continuity of the restricted map $\mu|_{[a,b]}$.

Indeed, fix $\varepsilon > 0$ and take $K_a \subseteq \mathcal{T}^{-1}(a)$ compact and such that $\mu_a(K_a) \geq 1 - \varepsilon$. Define $K := J^+(K_a) \cap J^-(\mathcal{T}^{-1}(b))$, which is compact on the strength of Proposition 2. Of course, $\text{supp } \mu_t \subseteq \mathcal{T}^{-1}(t) \subseteq J^-(\mathcal{T}^{-1}(b))$ for all $t \in [a, b]$ and hence

$$\mu_t(K) = \mu_t(J^+(K_a) \cap J^-(\mathcal{T}^{-1}(b))) = \mu_t(J^+(K_a)) \geq \mu_a(J^+(K_a)) = \mu_a(K_a) \geq 1 - \varepsilon,$$

where the inequality follows from (28) and Theorem 4. This completes the proof of tightness.

on the righthand side of (27) is well-defined, introduce a bounded Borel map $\Phi : C_{\mathcal{T}}^{[a,b]} \times C_{\mathcal{T}}^{[b,c]} \rightarrow \mathbb{R}$ via $\Phi(\gamma_1, \gamma_2) := F(\gamma_1 \sqcup \gamma_2)$ for $(\gamma_1, \gamma_2) \in \mathcal{Y}$ and zero otherwise. Observe that the integral can now be rewritten, by Fubini's theorem, as

$$\int_{\mathcal{M}} \left(\int_{\mathcal{Y}} (F \circ \sqcup) d(\sigma_1^x \times \sigma_2^x) \right) d\nu(x) = \int_{\mathcal{M}} \left(\int_{C_{\mathcal{T}}^{[b,c]}} \left(\int_{C_{\mathcal{T}}^{[a,b]}} \Phi(\gamma_1, \gamma_2) d\sigma_1^x(\gamma_1) \right) d\sigma_2^x(\gamma_2) \right) d\nu(x),$$

where the definition of the disintegration implies that the map $(x, \gamma_2) \mapsto \int_{C_{\mathcal{T}}^{[a,b]}} \Phi(\gamma_1, \gamma_2) d\sigma_1^x(\gamma_1)$ is Borel and bounded, and hence the same also concerns the map $x \mapsto \int_{C_{\mathcal{T}}^{[b,c]}} \left(\int_{C_{\mathcal{T}}^{[a,b]}} \Phi(\gamma_1, \gamma_2) d\sigma_1^x(\gamma_1) \right) d\sigma_2^x(\gamma_2)$.

We now move on to showing that $\lim_{s \rightarrow 0^+} \mu_{t+s} = \mu_t$ for any fixed $t \in [a, b)$ (for the other one-sided limit the proof is analogous).

Begin by fixing $\omega_{t,s} \in \Pi_c(\mu_t, \mu_{t+s})$ for each $s \in (0, b-t]$. For any $f \in C_c(\mathcal{M})$ one has

$$\left| \int_{\mathcal{M}} f d\mu_t - \int_{\mathcal{M}} f d\mu_{t+s} \right| = \left| \int_{\mathcal{M}^2} (f(p) - f(q)) d\omega_{t,s}(p, q) \right| \leq \int_{\text{supp } \omega_{t,s}} |f(p) - f(q)| d\omega_{t,s}(p, q). \quad (29)$$

Our aim is to prove that the rightmost integral becomes arbitrarily small for s sufficiently close to zero. To this end, let d_w denote the distance function associated with the complete Riemannian metric w introduced in (4) and let us first prove the following bound

$$\forall (p, q) \in \text{supp } \omega_{t,s} \quad d_w(p, q) \leq s \max_{r \in J^+(p) \cap J^-(q)} \sqrt{2u(r)\alpha(r)}. \quad (30)$$

Indeed, observe that

$$\text{supp } \omega_{t,s} \subseteq [\mathcal{T}^{-1}(t) \times \mathcal{T}^{-1}(t+s)] \cap J^+, \quad (31)$$

therefore $\mathcal{T}(p) = t$, $\mathcal{T}(q) = t+s$ and $p \preceq q$. Let thus $\gamma : [t, t+s] \rightarrow \mathcal{M}$ be a future-directed causal curve connecting p with q parametrized so as to make it an element of $C_{\mathcal{T}}^{[t, t+s]}$ (the existence of such γ is guaranteed by the definition of \preceq and by Proposition 4). Reasoning similarly as in the proof of Lemma 4, we obtain

$$\begin{aligned} d_w(p, q) &= d_w(\gamma(t), \gamma(t+s)) \leq c_\gamma \int_t^{t+s} \sqrt{2u(\gamma(\tau)) \cdot \alpha(\gamma(\tau))} d\tau \\ &\leq c_\gamma \max_{r \in J^+(p) \cap J^-(q)} \sqrt{2u(r)\alpha(r)} \cdot s, \end{aligned}$$

where noticing that $c_\gamma = (\mathcal{T}(q) - \mathcal{T}(p))/s = 1$ finishes the proof of (30). This bound, however, has the downside of being dependent on s and t (through p and q). It would be desirable to replace there the set $J^+(p) \cap J^-(q)$, over which the maximum is evaluated, with a compact set independent from p, q . One way of achieving this is to notice that when estimating the rightmost integral in (29), we can restrict to $(p, q) \notin (K_f^c)^2$, where $K_f := \text{supp } f$ is compact. We are thus looking for a *compact* superset of $\text{supp } \omega_{t,s} \setminus (K_f^c)^2$, which would be independent from s, t .

One possibility is

$$\mathcal{K} := K_f \times [J^+(K_f) \cap J^-(\mathcal{T}^{-1}(b))] \cup [J^-(K_f) \cap J^+(\mathcal{T}^{-1}(a))] \times K_f. \quad (32)$$

That such \mathcal{K} is compact follows from Proposition 2 *iii*). Furthermore, using (31), one can easily check that \mathcal{K} is indeed a superset of $\text{supp } \omega_{t,s} \setminus (K_f^c)^2$.

Ultimately, since $(p, q) \in \mathcal{K}$ implies that $J^+(p) \cap J^-(q) \subseteq J^+(\pi^1(\mathcal{K})) \cap J^-(\pi^2(\mathcal{K}))$ (notice the latter set is still compact by Proposition 2 *i*)), we obtain from (30) a somewhat modified bound

$$\forall (p, q) \in \text{supp } \omega_{t,s} \setminus (K_f^c)^2 \quad d_w(p, q) \leq s \max_{r \in J^+(\pi^1(\mathcal{K})) \cap J^-(\pi^2(\mathcal{K}))} \sqrt{2u(r)\alpha(r)}, \quad (33)$$

where the maximum is now clearly independent from both s and t .

Coming back to (29), we will show that

$$\lim_{s \rightarrow 0^+} \int_{\text{supp } \omega_{t,s} \setminus (K_f^c)^2} |f(p) - f(q)| d\omega_{t,s}(p, q) = 0. \quad (34)$$

To this end, fix $\varepsilon > 0$ and observe that (by the Heine–Borel theorem) $f \in C_c(\mathcal{M})$ is uniformly continuous, what means that there exists $\delta > 0$ such that $d_w(p, q) < \delta \Rightarrow |f(p) - f(q)| < \varepsilon$ for any $p, q \in \mathcal{M}$.

Let us thus consider $s < \delta \cdot \left(\max_{r \in J^+(\pi^1(\mathcal{K})) \cap J^-(\pi^2(\mathcal{K}))} \sqrt{2u(r)\alpha(r)} \right)^{-1}$. We obtain that

$$\int_{\text{supp } \omega_{t,s} \setminus (K_f^c)^2} |f(p) - f(q)| d\omega_{t,s}(p, q) < \varepsilon \int_{\text{supp } \omega_{t,s} \setminus (K_f^c)^2} d\omega_{t,s}(p, q) \leq \varepsilon,$$

what completes the proof that $\lim_{s \rightarrow 0^+} \mu_{t+s} = \mu_t$ for any $t \in [a, b]$. One similarly shows that $\lim_{s \rightarrow 0^-} \mu_{t+s} = \mu_t$ for any $t \in (a, b]$, and hence we obtain that the map μ is continuous on $[a, b]$. But the latter was an arbitrary compact subinterval of I , therefore μ is in fact continuous on entire I . \square

We are finally ready to prove the implication $i) \Rightarrow ii)$ of Theorem 5. We shall do it in four steps: first for $I = [a, b]$ (for some $a, b \in \mathbb{R}, a < b$), then for $I = [0, +\infty)$, and afterwards for $I = \mathbb{R}$, which altogether will imply the theorem's statement for any interval.

Proof of Theorem 5, $i) \Rightarrow ii)$.

Step 1. The $I = [a, b]$ case. The idea is to construct a sequence $(\sigma_n) \subseteq \mathcal{P}(C_{\mathcal{T}}^{[a,b]})$ such that $(\text{ev}_t)_{\#} \sigma_n = \mu_t$ for all t of the form $t_i^n := a + (b - a)i/2^n$, $i = 0, 1, 2, 3, \dots, 2^n$ and then show that it has a convergent subsequence, whose limit σ satisfies the above identity for *any* $t \in [a, b]$.

To this end, fix $n \in \mathbb{N}$ and for any $i = 1, \dots, 2^n$ let $S^i : J^+ \rightarrow C_{\mathcal{T}}^{[t_{i-1}^n, t_i^n]}$ denote the Borel map such that $(\text{ev}_{t_{i-1}^n} \times \text{ev}_{t_i^n}) \circ S^i = \text{id}_{J^+}$, existing by Proposition 10. Furthermore, for any $i = 1, \dots, 2^n$ let ω_i be a causal coupling of $\mu_{t_{i-1}^n}$ and $\mu_{t_i^n}$, existing by $i)$. Observe that because $\omega_i(J^+) = 1$, therefore $\omega_i|_{\mathcal{B}(J^+)} \in \mathcal{P}(J^+)$ for all $i = 1, \dots, 2^n$. All this allows us to define σ_n via multiple concatenation (Def. 4) as

$$\sigma_n := S_{\#}^1 (\omega_1|_{\mathcal{B}(J^+)}) \sqcup S_{\#}^2 (\omega_2|_{\mathcal{B}(J^+)}) \sqcup S_{\#}^3 (\omega_3|_{\mathcal{B}(J^+)}) \sqcup \dots \sqcup S_{\#}^{2^n} (\omega_{2^n}|_{\mathcal{B}(J^+)}) . \quad (35)$$

Observe that the right-hand side of (35) is unambiguous, because the operation \sqcup can be easily proven to be associative.

Notice now that $[\iota \circ (\text{ev}_a \times \text{ev}_b)]_{\#} \sigma_n$ is a causal coupling of μ_a and μ_b , what can be shown completely analogously as in the proof of the implication $ii) \Rightarrow i)$ in Section 2. By the arbitrariness of n , we thus obtain that $(\sigma_n) \subseteq [\iota \circ (\text{ev}_a \times \text{ev}_b)]_{\#}^{-1}(\Pi_c(\mu_a, \mu_b))$.

We now make the crucial observation: the set $[\iota \circ (\text{ev}_a \times \text{ev}_b)]_{\#}^{-1}(\Pi_c(\mu_a, \mu_b))$ is compact on the strength of Proposition 10, Lemma 9 and the obvious fact that $\iota : J^+ \hookrightarrow \mathcal{M}$ is a proper map. Therefore, (σ_n) has a subsequence that narrowly converges to certain $\sigma \in \mathcal{P}(C_{\mathcal{T}}^{[a,b]})$.

Observe that, by the very construction of the sequence (σ_n) , it is true that $(\text{ev}_{t_i^n})_{\#} \sigma = \mu_{t_i^n}$ for any $n \in \mathbb{N}$ and any $i = 0, 1, \dots, 2^n$. However, by Proposition 11, this already implies that $(\text{ev}_t)_{\#} \sigma = \mu_t$ is actually true for all $t \in [a, b]$.

Step 2. The $I = [0, +\infty)$ case. For the sake of convenience, for any $i \in \mathbb{N}$ denote $\mathcal{X}_i := C_{\mathcal{T}}^{[i-1, i]}(\mathcal{T}^{-1}(i-1), \mathcal{T}^{-1}(i))$ and let $\sigma_i \in \mathcal{P}(\mathcal{X}_i)$ satisfy $(\text{ev}_t)_{\#} \sigma_i = \mu_t$ for all $t \in [i-1, i]$. Now, for any $n \in \mathbb{N}$ define a measure $\sigma_n \in \mathcal{P}\left(\prod_{i=1}^n \mathcal{X}_i\right)$ recursively as

$$\sigma_1 := \sigma_1 \quad \text{and} \quad \forall n \in \mathbb{N} \quad \sigma_{n+1} := \int_{\mathcal{T}^{-1}(n)} (\sigma_n^x \times \sigma_{n+1}^x) d\mu_n(x), \quad (36)$$

where:

- $\{\sigma_n^x\}_{x \in \mathcal{T}^{-1}(n)}$ is the disintegration of σ_n w.r.t. the map $\text{ev}_n \circ \pi^n : \prod_{i=1}^n \mathcal{X}_i \rightarrow \mathcal{T}^{-1}(n)$;
- $\{\sigma_{n+1}^x\}_{x \in \mathcal{T}^{-1}(n)}$ is the disintegration of σ_{n+1} w.r.t. the map $\text{ev}_n : \mathcal{X}_{n+1} \rightarrow \mathcal{T}^{-1}(n)$.

For later use, we need to prove the following three properties of σ_n 's

$$\forall n \in \mathbb{N} \quad \pi_{\#}^n \sigma_{n+1} = \sigma_n \quad \text{and} \quad \pi_{\#}^n \sigma_n = \sigma_n \quad (37)$$

$$\forall n \in \mathbb{N} \quad \text{supp } \sigma_n \subseteq \left\{ (\gamma_i) \in \prod_{i=1}^n \mathcal{X}_i \mid \gamma_i(i) = \gamma_{i+1}(i), i = 1, \dots, n-1 \right\}, \quad (38)$$

where π^n denotes the canonical projection on the *first n arguments*¹³.

Identities (37,38) can be proven by a direct computation. Indeed, for any Borel map $F : \prod_{i=1}^n \mathcal{X}_i \rightarrow [0, +\infty]$ we have

$$\begin{aligned} \int_{\prod_{i=1}^n \mathcal{X}_i} F d(\pi_{\#}^n \sigma_{n+1}) &= \int_{\mathcal{T}^{-1}(n)} \left(\int_{\prod_{i=1}^{n+1} \mathcal{X}_i} F(\gamma_1, \dots, \gamma_n) d(\sigma_n^x \times \sigma_{n+1}^x)(\gamma_1, \dots, \gamma_n, \gamma_{n+1}) \right) d\mu_n(x) \\ &= \int_{\mathcal{T}^{-1}(n)} \left(\int_{\prod_{i=1}^n \mathcal{X}_i} F d\sigma_n^x \right) d\mu_n(x) = \int_{\prod_{i=1}^n \mathcal{X}_i} F d\sigma_n. \end{aligned}$$

Similarly, for any Borel map $f : \mathcal{X}_n \rightarrow [0, +\infty]$, assuming $n \geq 2$ (the case $n = 1$ is trivial),

$$\begin{aligned} \int_{\mathcal{X}_n} f d(\pi_{\#}^n \sigma_n) &= \int_{\mathcal{T}^{-1}(n-1)} \left(\int_{\prod_{i=1}^n \mathcal{X}_i} f(\gamma_n) d(\sigma_{n-1}^x \times \sigma_n^x)(\gamma_1, \dots, \gamma_{n-1}, \gamma_n) \right) d\mu_n(x) \\ &= \int_{\mathcal{T}^{-1}(n-1)} \left(\int_{\mathcal{X}_n} f d\sigma_n^x \right) d\mu_n(x) = \int_{\mathcal{X}_n} f d\sigma_n. \end{aligned}$$

We now proceed to proving (38) by induction over n .

For $n = 1$ there is nothing to prove. Assume then that (37) is proven up to certain n and suppose that $(\gamma_i) \in \text{supp } \sigma_{n+1}$, but nevertheless there exists $i_0 \in \{1, \dots, n\}$ such that $\gamma_{i_0}(i_0) \neq \gamma_{i_0+1}(i_0)$.

If $i_0 < n$, then $\pi^n((\gamma_i)) \notin \text{supp } \sigma_n$ by the induction hypothesis. On the other hand, using (37) one can easily show that $\pi^n(\text{supp } \sigma_{n+1}) \subseteq \text{supp } \sigma_n$ and we obtain a contradiction.

¹³Not to be confused with π^n , which denotes the canonical projection on the n -th coordinate.

Thus, the only remaining possibility is $i_0 = n$. Denote $p := \gamma_n(n)$ and $q := \gamma_{n+1}(n)$. By assumption $p \neq q$ and so there exist their open neighborhoods $U_p, U_q \subseteq \mathcal{M}$, which are disjoint. Define an open neighborhood of (γ_i) in $\prod_{i=1}^{n+1} \mathcal{X}_i$ via

$$\mathcal{U} := (\text{ev}_n \circ \pi^n)^{-1}(U_p) \times \text{ev}_n^{-1}(U_q)$$

By assumption, $\sigma_{n+1}(\mathcal{U}) > 0$. On the other hand, one has that

$$(\sigma_n^x \times \sigma_{n+1}^x)(\mathcal{U}) = 0 \quad \text{for any } x \in \mathcal{T}^{-1}(n) \text{ for which it is defined.} \quad (39)$$

Indeed, by the disintegration theorem, for any x as specified above,

$$\text{supp}(\sigma_n^x \times \sigma_{n+1}^x) \subseteq \text{supp} \sigma_n^x \times \text{supp} \sigma_{n+1}^x \subseteq (\text{ev}_n \circ \pi^n)^{-1}(x) \times \text{ev}_n^{-1}(x),$$

and observe that the rightmost set is disjoint with \mathcal{U} , as otherwise we would have $x \in U_p \cap U_q$. By (36), this means that $\sigma_{n+1}(\mathcal{U}) = 0$, hence a contradiction.

Coming back to the main course of the proof, we now invoke the Kolmogorov extension theorem as given in [3]. Suppose one is given a family of Polish spaces $\{\mathcal{X}_i\}_{i \in \mathbb{N}}$ and a family of measures $\{\sigma_n\}_{n \in \mathbb{N}}$ such that $\sigma_n \in \mathcal{P}(\prod_{i=1}^n \mathcal{X}_i)$ and satisfying $\pi_{\#}^n \sigma_{n+1} = \sigma_n$ (which, by (36,37), is the case here). Then there exists $\sigma_{\infty} \in \mathcal{P}(\prod_{i=1}^{\infty} \mathcal{X}_i)$ such that $\pi_{\#}^n \sigma_{\infty} = \sigma_n$.

Of course, σ_{∞} is *not* the desired measure $\sigma \in \mathcal{P}(C_{\mathcal{T}}^{[0,+\infty)})$, however we will now construct the latter from the former. To this end, define the map $H : C_{\mathcal{T}}^{[0,+\infty)} \rightarrow \prod_{i=1}^{\infty} \mathcal{X}_i$ via

$$\forall \gamma \in C_{\mathcal{T}}^{[0,+\infty)} \quad H(\gamma) := (\gamma|_{[i-1, i]})_{i \in \mathbb{N}} \quad (40)$$

Observe that the image $H(C_{\mathcal{T}}^{[0,+\infty)})$ is the set of those sequences $(\gamma_i) \in \prod_{i=1}^{\infty} \mathcal{X}_i$ which satisfy $\gamma_i(i) = \gamma_{i+1}(i)$ for *all* $i \in \mathbb{N}$.

Clearly, H is one-to-one and hence it has an inverse¹⁴

$$H^{-1} : H(C_{\mathcal{T}}^{[0,+\infty)}) \rightarrow C_{\mathcal{T}}^{[0,+\infty)}, \quad H^{-1}((\gamma_i))(t) = \gamma_{[t]+1}(t) \quad (41)$$

for any $t \geq 0$. We claim that both H and H^{-1} are *continuous* maps.

To prove this claim, let us denote $\gamma := (\gamma_i) \in \prod_{i=1}^{\infty} \mathcal{X}_i$ and recall that a sequence $(\gamma_n) \subseteq \prod_{i=1}^{\infty} \mathcal{X}_i$ is convergent in the product topology iff for every $i \in \mathbb{N}$ the sequence $(\pi^i(\gamma_n)) \subseteq \mathcal{X}_i$ is convergent. Now, take any $(\gamma_n) \subseteq H(C_{\mathcal{T}}^{[0,+\infty)})$ convergent in the product topology, and consider a sequence $(H^{-1}(\gamma_n)) \subseteq C_{\mathcal{T}}^{[0,+\infty)}$. Taking any compact subset of $[0, +\infty)$, we can cover it with finitely many intervals of the form $[i-1, i]$, on all of which we have the uniform convergence of sequences $(H^{-1}(\gamma_n)|_{[i-1, i]})$ and hence $(H^{-1}(\gamma_n))$ converges in the compact-open topology. Conversely, if $(\gamma_n) \subseteq C_{\mathcal{T}}^{[0,+\infty)}$ converges in the compact-open topology, then $(\gamma_n|_{[i-1, i]})$ converges in \mathcal{X}_i for every $i \in \mathbb{N}$ and hence $(H(\gamma_n))$ converges in the product topology.

¹⁴Intuitively, it can be regarded as a simultaneous concatenation of countably many causal curves.

Notice that the above reasoning proves in particular that the image of H is closed in $\prod_{i=1}^{\infty} \mathcal{X}_i$ and as such it is a Polish space.

To finish the proof, we want to define $\sigma := (H^{-1})_{\#} \sigma_{\infty}$, however we do not a priori know whether σ_{∞} is concentrated on the image of H , i.e. whether

$$\text{supp } \sigma_{\infty} \subseteq H \left(C_{\mathcal{T}}^{[0,+\infty)} \right) = \left\{ (\gamma_i) \in \prod_{i=1}^{\infty} \mathcal{X}_i \mid \gamma_i(i) = \gamma_{i+1}(i), i \in \mathbb{N} \right\}. \quad (42)$$

This is, in a sense, the “ $n \rightarrow +\infty$ ” version of property (38) and the reasoning is somewhat similar. Concretely, suppose on the contrary that one can find $\gamma = (\gamma_i) \in \text{supp } \sigma_{\infty}$, for which there exists $i_0 \in \mathbb{N}$ such that $\gamma_{i_0}(i_0) \neq \gamma_{i_0+1}(i_0)$. This property will be preserved if we truncate γ to $\pi^n(\gamma)$ for any fixed $n \geq i_0 + 1$ and therefore $\pi^n(\gamma) \notin \text{supp } \sigma_n$. On the other hand, using the property that $\pi^n_{\#} \sigma_{\infty} = \sigma_n$ guaranteed by the Kolmogorov theorem, one can show that $\pi^n(\text{supp } \sigma_{\infty}) \subseteq \text{supp } \sigma_n$ and thus we arrive at a contradiction.

All in all, we finally have a well-defined $\sigma \in \mathcal{P}(C_{\mathcal{T}}^{[0,+\infty)})$. One can still be anxious whether it really inherits the property $(\text{ev}_t)_{\#} \sigma = \mu_t$ ($t \geq 0$) from its “constituents” σ_i ’s. This can be in fact checked directly by first noticing that, by formula (41),

$$\forall t \geq 0 \quad \text{ev}_t \circ H^{-1} = \text{ev}_t \circ \pi^{[t]+1} = \text{ev}_t \circ \pi^{[t]+1} \circ \pi^{[t]+1},$$

and then by applying this observation as follows:

$$(\text{ev}_t)_{\#} \sigma = (\text{ev}_t \circ H^{-1})_{\#} \sigma_{\infty} = (\text{ev}_t \circ \pi^{[t]+1} \circ \pi^{[t]+1})_{\#} \sigma_{\infty} = (\text{ev}_t)_{\#} \sigma_{[t]+1} = \mu_t$$

for any $t \geq 0$, where we have used both properties (37,38).

Step 3. The $I = \mathbb{R}$ case. Let $\sigma_+ \in \mathcal{P}(C_{\mathcal{T}}^{[0,+\infty)})$ denote the measure constructed in Step 2. One can similarly construct $\sigma_- \in \mathcal{P}(C_{\mathcal{T}}^{(-\infty,0]})$ such that $(\text{ev}_t)_{\#} \sigma_- = \mu_t$ for all $t \leq 0$. To this end, define $\mathcal{X}_i := C_{\mathcal{T}}^{[-i, -i+1]}(\mathcal{T}^{-1}(-i), \mathcal{T}^{-1}(-i+1))$ and proceed as before, applying some minor modifications reflecting the fact that here, while constructing σ_{∞} , we are moving towards the *lower* values of \mathcal{T} .

On the strength of Remark 5, one can define $\sigma := \sigma_- \sqcup \sigma_+$, and Remark 6 guarantees that $(\text{ev}_t)_{\#} \sigma = \mu_t$ for all $t \in \mathbb{R}$.

Step 4. The general case. The arguments used in the previous two steps can be easily adapted to any kind of a (nonempty) interval I . More concretely, for any $a, b \in \mathbb{R}$ and $a < b$:

- For $I = [a, +\infty)$ apply the reasoning from Step 2. with

$$\mathcal{X}_i := C_{\mathcal{T}}^{[a+i-1, a+i]}(\mathcal{T}^{-1}(a+i-1), \mathcal{T}^{-1}(a+i))$$

and other minor modifications that are necessary.

- For $I = [a, b)$ apply the reasoning from Step 2. with

$$\mathcal{X}_i := C_{\mathcal{T}}^{[b+(a-b)2^{-i}, b+(a-b)2^{-i-1}]}(\mathcal{T}^{-1}(b+(a-b)2^{-i}), \mathcal{T}^{-1}(b+(a-b)2^{-i-1}))$$

and other minor modifications that are necessary.

- For $I = (-\infty, b]$ or $I = (-a, b]$ modify the above cases in a similar spirit as Step 3. modified Step 2. in order to construct σ_- .
- Finally, for $I = (a, b)$, $I = (a, +\infty)$ or $I = (-\infty, b)$, simply take suitable pairs of σ 's provided by the previous cases and concatenate them, similarly as in Step 3.

□

5 Physical content of Theorem 5

Let us discuss the physical reason behind calling a measure-valued map *causal* in the sense introduced in Theorem 5 as well as the physical content of the theorem itself.

Suppose physicist **A** wants to describe the time-evolution of a physical quantity Q distributed in space — be it a mass or charge distribution — whose total amount is conserved. Assume first that the background spacetime is Minkowski. In order to describe a dynamical phenomenon, the physicist more or less implicitly chooses the time parameter t and thus employs a particular foliation of the Minkowski spacetime by t -slices — the hypersurfaces of simultaneity. In **A**'s description, the distribution of Q at the time instant t might be modelled by a measure $\mu_t \in \mathcal{P}(\mathcal{M})$ (we normalize the total amount of Q to one) supported on the corresponding t -slice.

A is interested in whether the time evolution $t \mapsto \mu_t$ does not violate Einstein's causality, understood as the impossibility of superluminal propagation of any physical object or interaction. In case of Q , which is a quantity distributed in space, the velocity bound concerns the “infinitesimal portions” of Q (also called *parcels*). Intuitively, the distribution of Q evolves causally iff each of its parcels travels along a causal curve.

The above intuition is made mathematically rigorous in Definition 3 (see [13] for a more detailed exposition) which, when applied to the family of measures in **A**'s description, produces exactly what we called in Theorem 5 a causal measure-valued map (with \mathcal{T} defined as yielding the t -coordinate of its argument). Consult [14] for more detailed discussion and physical examples.

In this setting, what Theorem 5 says is that **A** can provide an alternative description of the dynamical phenomenon in question. Concretely, instead of using a causal measure-valued map $t \mapsto \mu_t$, **A** can use a single probability measure σ living on the suitable Polish space of causal curves. Both descriptions are equivalent, i.e. they contain exactly the same information.

Let us emphasize that Theorem 5 works not only in the Minkowski spacetime, but in any globally hyperbolic spacetime \mathcal{M} . In order to describe the time-evolution of Q , the physicist **A** chooses a Cauchy temporal function \mathcal{T}_1 and uses the Geroch–Bernal–Sánchez (GBS) splitting (denoted by Φ in Theorem 1) and thus picks a particular foliation of \mathcal{M} by the Cauchy hypersurfaces comprising the level sets of \mathcal{T}_1 . In this case, every instantaneous distribution of Q is modelled by a measure μ_t supported on the corresponding level set $\mathcal{T}_1^{-1}(t)$.

Even in the Minkowski case, however, the following questions arise: Suppose that another physicist **B** would like to describe *the same* dynamical phenomenon, but choosing a *different* Cauchy temporal function \mathcal{T}_2 and thus a different foliation of \mathcal{M} . Of course, he would obtain a completely different family of measures, living on Cauchy hypersurfaces transversal to those employed by **A**. The questions are: What is the relationship between

A's and **B's** descriptions? Are there any invariants within their descriptions? Would they always agree about the causality of their measure-valued maps?

In order to answer these questions, we will need a special variant of Theorem 5, involving yet another Polish space of causal curves, defined below.

Definition 5. Let \mathcal{M} be a globally hyperbolic spacetime and let $\mathcal{T} : \mathcal{M} \rightarrow \mathbb{R}$ be a Cauchy time function. Define the space $\mathcal{I}_{\mathcal{T}}$ via

$$\mathcal{I}_{\mathcal{T}} := \{\gamma \in C_{\mathcal{T}}^{\mathbb{R}} \mid \mathcal{T} \circ \gamma = \text{id}_{\mathbb{R}}\}$$

endowed with the topology induced from $C_{\mathcal{T}}^{\mathbb{R}}$ (i.e. the topology of uniform convergence on compact subsets of \mathbb{R}).

Remark 7. Recall that $\text{ev}_t : C_{\mathcal{T}}^{\mathbb{R}} \rightarrow \mathbb{R}$ denotes the evaluation map for any $t \in \mathbb{R}$. Observe that $\mathcal{I}_{\mathcal{T}} = (\mathcal{T} \circ \text{ev}_0)^{-1}(0) \cap (\mathcal{T} \circ \text{ev}_1)^{-1}(1)$. Indeed, the inclusion \subseteq is trivial, whereas to prove \supseteq , assume that $\gamma \in C_{\mathcal{T}}^{\mathbb{R}}$ satisfies $\mathcal{T}(\gamma(0)) = 0$ and $\mathcal{T}(\gamma(1)) = 1$. Using (7), we obtain that

$$\forall t \in \mathbb{R} \quad \mathcal{T}(\gamma(t)) = (1-t)\mathcal{T}(\gamma(0)) + t\mathcal{T}(\gamma(1)) = t$$

and hence $\gamma \in \mathcal{I}_{\mathcal{T}}$. As a side note, observe that $c_{\gamma} = 1$.

The above remark makes it clear that $\mathcal{I}_{\mathcal{T}}$ is closed in $C_{\mathcal{T}}^{\mathbb{R}}$ and hence it is a Polish space. The letter \mathcal{I} was chosen to stand for “inextendible”, due to the following fact.

Proposition 12. Let \mathcal{M} and \mathcal{T} be as above and let $\mathcal{C}_{\text{inext}}$ denote the set of all inextendible causal paths on \mathcal{M} . Then the map $[\cdot] : \mathcal{I}_{\mathcal{T}} \rightarrow \mathcal{C}_{\text{inext}}, \gamma \mapsto [\gamma]$ is a bijection.

Proof That the map $[\cdot]$ is well defined follows from the proof of Proposition 8. Surjectivity is a consequence of Proposition 7 (the last bullet), where in the formula for λ we choose $A = 1$ and $B = 0$.

In order to prove injectivity, assume that $[\gamma_1] = [\gamma_2]$ for some $\gamma_1, \gamma_2 \in \mathcal{I}_{\mathcal{T}}$. This means that there exists $\lambda \in C(\mathbb{R}, \mathbb{R})$ such that $\gamma_1 \circ \lambda = \gamma_2$. However, composing both sides of the last identity with \mathcal{T} , by the very definition of $\mathcal{I}_{\mathcal{T}}$ one obtains that

$$\lambda = \text{id}_{\mathbb{R}} \circ \lambda = \mathcal{T} \circ \gamma_1 \circ \lambda = \mathcal{T} \circ \gamma_2 = \text{id}_{\mathbb{R}}$$

and so $\gamma_1 = \gamma_2$. □

Using the above bijection, we propose how to topologize $\mathcal{C}_{\text{inext}}$. Namely, we transport the Polish space topology from $\mathcal{I}_{\mathcal{T}}$ using the bijective map $[\cdot]$. Of course, such a topology might a priori depend on \mathcal{T} , which would make it rather artificial. However, this turns out not to be the case — at least as far as temporal functions are concerned.

Proposition 13. Let \mathcal{M} be a globally hyperbolic spacetime and let $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{M} \rightarrow \mathbb{R}$ be Cauchy temporal functions. Then the map $\tilde{\cdot} : \mathcal{I}_{\mathcal{T}_1} \rightarrow \mathcal{I}_{\mathcal{T}_2}$, defined via (14), that is

$$\forall \gamma \in \mathcal{I}_{\mathcal{T}_1} \quad \tilde{\gamma} := \gamma \circ (\mathcal{T}_2 \circ \gamma)^{-1} \circ \mathcal{T}_1 \circ \gamma \tag{43}$$

is a homeomorphism.

Proof Proposition 9 guarantees that $\tilde{\gamma} \in C_{\mathcal{T}_2}^{\mathbb{R}}$ for any $\gamma \in \mathcal{I}_{\mathcal{T}_1}$. However, noticing that $\mathcal{T}_2 \circ \tilde{\gamma} = \mathcal{T}_1 \circ \gamma = \text{id}_{\mathbb{R}}$ we have that in fact $\tilde{\gamma} \in \mathcal{I}_{\mathcal{T}_2}$ and so the map $\sim : \mathcal{I}_{\mathcal{T}_1} \rightarrow \mathcal{I}_{\mathcal{T}_2}$ is well defined. Swapping \mathcal{T}_1 and \mathcal{T}_2 , one similarly proves that the inverse of \sim is well defined as well.

That \sim and its inverse are continuous follows from Theorem 3 and from elementary properties of the subspace topology. \square

Proposition 13 guarantees that no matter which Cauchy temporal function \mathcal{T} we choose, transporting the topology from $\mathcal{I}_{\mathcal{T}}$ onto $\mathcal{C}_{\text{inext}}$ always yields the same topology on the latter. Leaving the study of this topology for another occasion, we now state the variant of Theorem 5.

Theorem 7. *Let \mathcal{M} be a globally hyperbolic spacetime and let $\mathcal{T} : \mathcal{M} \rightarrow \mathbb{R}$ be a Cauchy temporal function. For any map $\mu : \mathbb{R} \rightarrow \mathcal{P}(\mathcal{M})$, $t \mapsto \mu_t$ the following conditions are equivalent*

i) *The map μ satisfies $\text{supp } \mu_t \subseteq \mathcal{T}^{-1}(t)$ for every $t \in \mathbb{R}$ and is causal, i.e. $s \leq t$ implies $\mu_s \leq \mu_t$ for all $s, t \in \mathbb{R}$.*

ii) *There exists $v \in \mathcal{P}(\mathcal{I}_{\mathcal{T}})$ such that¹⁵ $(\text{ev}_t|_{\mathcal{I}_{\mathcal{T}}})_{\#}v = \mu_t$ for every $t \in \mathbb{R}$.*

Proof i) \Rightarrow ii) Theorem 5 guarantees the existence of $\sigma \in \mathcal{P}(C_{\mathcal{T}}^{\mathbb{R}})$ satisfying $(\text{ev}_t)_{\#}\sigma = \mu_t$ for all $t \in \mathbb{R}$. Let $\iota : \mathcal{I}_{\mathcal{T}} \hookrightarrow C_{\mathcal{T}}^{\mathbb{R}}$ denote the canonical topological embedding. Suppose we have shown that $\sigma(\mathcal{I}_{\mathcal{T}}) = 1$. Then, defining v via $v(E) := \sigma(\iota(E))$ for any $E \in \mathcal{B}(\mathcal{I}_{\mathcal{T}})$ one would obtain an element of $\mathcal{P}(\mathcal{I}_{\mathcal{T}})$ with the desired properties. Indeed, for any $t \in \mathbb{R}$ and any $\mathcal{X} \in \mathcal{B}(\mathcal{M})$ we would have

$$(\text{ev}_t|_{\mathcal{I}_{\mathcal{T}}})_{\#}v(\mathcal{X}) = (\text{ev}_t \circ \iota)_{\#}v(\mathcal{X}) = (\text{ev}_t)_{\#}v(\iota^{-1}(\mathcal{X})) = (\text{ev}_t)_{\#}\sigma(\mathcal{X}) = \mu_t(\mathcal{X}).$$

Hence, we only need to show that $\sigma(\mathcal{I}_{\mathcal{T}}) = 1$. To this end, observe first that

$$\forall t \in \mathbb{R} \quad \sigma((\mathcal{T} \circ \text{ev}_t)^{-1}(t)) = (\mathcal{T} \circ \text{ev}_t)_{\#}\sigma(\{t\}) = \mathcal{T}_{\#}\mu_t(\{t\}) = \mu_t(\mathcal{T}^{-1}(t)) = 1, \quad (44)$$

where the last equality follows from $\text{supp } \mu_t \subseteq \mathcal{T}^{-1}(t)$.

Remark 7, the inclusion-exclusion principle and (44) allow to obtain

$$\begin{aligned} \sigma(\mathcal{I}_{\mathcal{T}}) &= \sigma((\mathcal{T} \circ \text{ev}_0)^{-1}(0) \cap (\mathcal{T} \circ \text{ev}_1)^{-1}(1)) \\ &= \underbrace{\sigma((\mathcal{T} \circ \text{ev}_0)^{-1}(0))}_{=1} + \underbrace{\sigma((\mathcal{T} \circ \text{ev}_1)^{-1}(1))}_{=1} - \underbrace{\sigma((\mathcal{T} \circ \text{ev}_0)^{-1}(0) \cup (\mathcal{T} \circ \text{ev}_1)^{-1}(1))}_{=1} = 1. \end{aligned}$$

ii) \Rightarrow i) Define $\sigma := \iota_{\#}v \in \mathcal{P}(C_{\mathcal{T}}^{\mathbb{R}})$ and observe that for any $t \in \mathbb{R}$ one has

$$(\text{ev}_t)_{\#}\sigma = (\text{ev}_t \circ \iota)_{\#}v = (\text{ev}_t|_{\mathcal{I}_{\mathcal{T}}})_{\#}v = \mu_t.$$

Moreover, $\text{supp } \mu_t \subseteq \mathcal{T}^{-1}(t)$ for all $t \in \mathbb{R}$, because

$$\mu_t(\mathcal{T}^{-1}(t)) = \mathcal{T}_{\#}\mu_t(\{t\}) = (\mathcal{T} \circ \text{ev}_t|_{\mathcal{I}_{\mathcal{T}}})_{\#}v(\{t\}) = v((\mathcal{T} \circ \text{ev}_t|_{\mathcal{I}_{\mathcal{T}}})^{-1}(t)) = v(\mathcal{I}_{\mathcal{T}}) = 1,$$

where the penultimate equality follows by the very definition of $\mathcal{I}_{\mathcal{T}}$. We can thus apply Theorem 5 and conclude that the map $t \mapsto \mu_t$ must be causal. \square

¹⁵By definition, the domain of ev_t is $C_{\mathcal{T}}^{\mathbb{R}}$ and so we have to restrict it to $\mathcal{I}_{\mathcal{T}}$ in order to push forward the measure v .

We now return to the question of the relationship between **A**'s and **B**'s descriptions of a dynamical phenomenon.

Let the map $I_1 \ni t \mapsto \mu_t$, $\text{supp } \mu_t \subseteq \mathcal{T}_1^{-1}(t)$ be the time-evolution of a physical quantity Q as described by physicist **A** employing the Cauchy temporal function \mathcal{T}_1 . Similarly, the map $I_2 \ni \tau \mapsto \nu_\tau$, $\text{supp } \nu_\tau \subseteq \mathcal{T}_2^{-1}(\tau)$ will denote the time-evolution of Q according to the description of physicist **B** who uses the Cauchy temporal function \mathcal{T}_2 . Since we assume that Q is conserved in the course of evolution, we take $I_1, I_2 = \mathbb{R}$. Albeit fragmentary descriptions in which the dynamical parameter $t \in I_1 \subsetneq \mathbb{R}$ are practically useful (cf. [14] and references therein), they unavoidably distinguish at least one Cauchy hypersurface (e.g. $\mathcal{T}_1^{-1}(\inf I_1)$ if I_1 were bounded from below) and so they are trivially not GBS-splitting-independent.

Assume that both maps $t \mapsto \mu_t$ and $\tau \mapsto \nu_\tau$ are causal. Because $I_1, I_2 = \mathbb{R}$, both **A** and **B** can apply Theorem 7 and encapsulate their descriptions within the measures $v_i \in \mathcal{P}(\mathcal{I}_{\mathcal{T}_i})$, $i = 1, 2$, respectively. Recall that Proposition 13 provides a homeomorphism $\sim : \mathcal{I}_{\mathcal{T}_1} \rightarrow \mathcal{I}_{\mathcal{T}_2}$, which is a suitable reparametrization map. Let \tilde{v}_1 denote the pushforward of v_1 by \sim . We argue that $v_2 = \tilde{v}_1$.

The physical reasons for this equality to be true can be explained as follows. Imagine **A** and **B** want to describe the motion of a pointlike particle. They are given its unparametrized world line – a single inextendible causal path $[\gamma] \in \mathcal{C}_{\text{inext}}$. Physicist **A** parametrizes $[\gamma]$ by assigning to each $p \in [\gamma]$ the number $\mathcal{T}_1(p)$, obtaining the curve $\gamma_1 \in \mathcal{I}_{\mathcal{T}_1}$. Similarly, physicist **B** obtains the curve $\gamma_2 \in \mathcal{I}_{\mathcal{T}_2}$ assigning to each $p \in [\gamma]$ the number $\tau := \mathcal{T}_2(p)$.

We claim that $\gamma_2 = \tilde{\gamma}_1$. Indeed, denote by $[\cdot]_i : \mathcal{I}_{\mathcal{T}_i} \rightarrow \mathcal{C}_{\text{inext}}$, $i = 1, 2$, the bijections discussed in Proposition 12. We obviously have

$$[\gamma_2]_2 = [\gamma] = [\gamma_1]_1 = [\tilde{\gamma}_1]_2,$$

where the last equality is true because $\tilde{\gamma}_1$ is a reparametrization of γ_1 . The claim follows from the injectivity of $[\cdot]_2$.

In other words, it is precisely the homeomorphism $\sim : \mathcal{I}_{\mathcal{T}_1} \rightarrow \mathcal{I}_{\mathcal{T}_2}$, which allows to switch between **A**'s and **B**'s parametrizations of any given inextendible causal path. Via its pushforward map, the above argumentation extends onto *measures on the spaces of causal curves* and we conclude that $v_2 = \tilde{v}_1$.

We can further exploit the above reasoning with the aid of the Polish space structure we endowed $\mathcal{C}_{\text{inext}}$ with. Namely, for $i = 1, 2$, let $[v_i]_i \in \mathcal{P}(\mathcal{C}_{\text{inext}})$ denote the pushforward of v_i by the bijection $[\cdot]_i$, which is a homeomorphism by the very definition of the topology on $\mathcal{C}_{\text{inext}}$. Because for any $\gamma_1 \in \mathcal{I}_{\mathcal{T}_1}$ we know that $[\tilde{\gamma}_1]_2 = [\gamma_1]_1$, therefore for measures we obtain $[v_2]_2 = [\tilde{v}_1]_2 = [v_1]_1$.

We have thus obtained an invariant implicitly contained in both **A**'s and **B**'s descriptions. It is the measure on the space $\mathcal{C}_{\text{inext}}$ of inextendible causal curves, and as such it does not pertain to any particular GBS splitting. To put it differently: Just as the path $[\gamma] \in \mathcal{C}_{\text{inext}}$ is the GBS-splitting-independent spatiotemporal object modelling the motion of a pointlike particle, one can analogously say that the measure $[v] \in \mathcal{P}(\mathcal{C}_{\text{inext}})$ is the GBS-splitting-independent spatiotemporal object which models the dynamics of a physical quantity Q distributed in space.

With the above in mind, we now address the question whether **A** and **B** would always agree on the causality of their measure-valued maps. To begin with, suppose that **A**'s map $t \mapsto \mu_t$ is causal. Therefore, **A** can rephrase his/her description in the form of $v_1 \in \mathcal{P}(\mathcal{I}_{\mathcal{T}_1})$ and then in the form of $[v_1] =: [v] \in \mathcal{C}_{\text{inext}}$ which is independent of any particular choice

of Cauchy temporal function and its associated GBS splitting. Any other physicist \mathbf{B} , describing the same dynamical phenomenon, accesses the same spatiotemporal object $[v]$, but he/she does so by means of a GBS splitting associated to a different Cauchy temporal function \mathcal{T}_2 . More concretely, \mathbf{B} applies to $[v]$ the inverse map $[\cdot]_2^{-1}$, which “parametrizes” $[v]$ so that it becomes $v_2 \in \mathcal{P}(\mathcal{I}_{\mathcal{T}_2})$. The existence of the latter, by the implication $ii) \Rightarrow i)$ in Theorem 7, assures that \mathbf{B} ’s map $\tau \mapsto \nu_\tau$ must be causal as well.

Summarizing the above discussion, we are allowed to say that the notion of a causal measure-valued map $t \mapsto \mu_t$ such that $\text{supp } \mu_t \subseteq \mathcal{T}^{-1}(t)$, is a correct way of modelling the dynamics of a spatially distributed physical quantity. Even though it seems tightly associated with a concrete choice of the GBS splitting, it is mathematically equivalent to a well-defined splitting-independent spatiotemporal object — the measure $[v] \in \mathcal{P}(\mathcal{C}_{\text{next}})$.

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